

QUASI-SYMMETRIES OF DETERMINANTAL POINT PROCESSES.

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ABSTRACT. The main result of this paper is that determinantal point processes on \mathbb{R} corresponding to projection operators with integrable kernels are quasi-invariant, in the continuous case, under the group of diffeomorphisms with compact support (Theorem 1.5); in the discrete case, under the group of all finite permutations of the phase space (Theorem 1.7). The Radon-Nikodym derivative is computed explicitly and is given by a regularized multiplicative functional. Theorem 1.5 applies, in particular, to the sine-process, as well as to determinantal point processes with the Bessel and the Airy kernels; Theorem 1.7 to the discrete sine-process and the Gamma kernel process. The paper answers a question of Grigori Olshanski.

1. INTRODUCTION

G. Olshanski [18] established the quasi-invariance of the determinantal measure corresponding to the Gamma kernel under the group of finite permutations of \mathbb{Z} and expressed the Radon-Nikodym derivative as a multiplicative functional. S. Ghosh and Y. Peres [9], [10] showed, for the Ginibre ensemble and the Gaussian zero process, that the conditional distribution of particles in a bounded domain, with the configuration fixed in the exterior, is equivalent to the Lebesgue measure.

This paper is mainly concerned with determinantal point processes on \mathbb{R} governed by kernels admitting an *integrable* representation

$$\frac{A(x)B(y) - B(x)A(y)}{x - y}.$$

Under some additional assumptions it is proved that the measure class of the corresponding determinantal measures is preserved, in the continuous case, under the group of diffeomorphisms with compact support (Theorem 1.5); in the discrete case, under the group of finite permutations of the phase space (Theorem 1.7). The key step in the proof is the equivalence of reduced Palm measures corresponding to different l -tuples of points (p_1, \dots, p_l) ,

(q_1, \dots, q_l) in the phase space; the corresponding Radon-Nikodym derivative is the regularized multiplicative functional corresponding to the function

$$(1) \quad \frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)}.$$

The Radon-Nikodym derivative thus has the same form for all the processes with integrable kernels; the normalizing constants do, of course, depend on the specific process.

Olshanski [18] proves the quasi-invariance of the Gamma-kernel process by a limit transition from finite-dimensional approximations. The argument in this paper is direct: first, it is shown that the Palm subspaces corresponding to conditioning at points p and q are taken one to the other by multiplication by the function $(x - p)/(x - q)$; after which, the proof is completed using a general result of [6] (see also [5]) that multiplying the range of the projection operator Π inducing a determinantal measure \mathbb{P}_Π by a function g , corresponds, under certain additional assumptions, to taking the product of the determinantal measure \mathbb{P}_Π by the multiplicative functional Ψ_g induced by the function g .

1.1. Quasi-symmetries of the sine-process. For example, let

$$\mathcal{S}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}, x, y \in \mathbb{R}$$

be the standard sine-kernel on \mathbb{R} , and let $\mathbb{P}_\mathcal{S}$ be the sine-process, the determinantal measure induced by the kernel \mathcal{S} on the space $\text{Conf}(\mathbb{R})$ of configurations on \mathbb{R} .

Proposition 1.1. *The measure $\mathbb{P}_\mathcal{S}$ is quasi-invariant under the group of diffeomorphisms of \mathbb{R} with compact support.*

To write down the Radon-Nikodym derivative, we need more notation. Given a Borel measure μ on a Borel space \mathcal{X} and a Borel automorphism T of \mathcal{X} , denote by $\mu \circ T$ the measure defined by $\mu \circ T(\mathcal{Z}) = \mu(T(\mathcal{Z}))$ for all Borel subsets $\mathcal{Z} \subset \mathcal{X}$. Since T is invertible, the measure $\mu \circ T$ is well-defined, and, for any μ -integrable Borel function f on \mathcal{X} , satisfies

$$\int_{\mathcal{X}} f \circ T d\mu \circ T = \int_{\mathcal{X}} f d\mu$$

as well as $(f\mu) \circ T = (f \circ T)(\mu \circ T)$.

A diffeomorphism F of \mathbb{R} acts on $\text{Conf}(\mathbb{R})$ by sending a configuration X to the configuration $F(X) = \{F(x), x \in X\}$; slightly abusing notation, we keep the same symbol F for this induced action.

By definition, the measure $\mathbb{P}_\mathcal{S} \circ F$ is determinantal with kernel

$$F^*\mathcal{S}(x, y) = \sqrt{F'(x)F'(y)}\mathcal{S}(F(x), F(y)).$$

If $L_{\mathcal{S}}$ is the range in $L_2(\mathbb{R})$ of the projection operator $\Pi_{\mathcal{S}}$ with kernel \mathcal{S} , then the kernel $F^*\mathcal{S}$ induces the operator $\Pi_{F^*\mathcal{S}}$ of orthogonal projection onto the subspace

$$L_{F^*\mathcal{S}} = F_*L_{\mathcal{S}} = \{\sqrt{F'} \cdot \varphi \circ F, \varphi \in L\}.$$

By definition of the sine-kernel, the spaces $L_{\mathcal{S}}$ and $L_{F^*\mathcal{S}}$ only consist of continuous functions; given l distinct points $q_1, \dots, q_l \in \mathbb{R}$, we set

$$L_{\mathcal{S}}(q_1, \dots, q_l) = \{\varphi \in L : \varphi(q_1) = \dots = \varphi(q_l) = 0\};$$

and we denote $\Pi_{\mathcal{S}}^{q_1, \dots, q_l}$ the operator of orthogonal projection onto the subspace $L_{\mathcal{S}}(q_1, \dots, q_l)$. The subspace $L_{F^*\mathcal{S}}(q_1, \dots, q_l)$ and the operator $\Pi_{F^*\mathcal{S}}^{q_1, \dots, q_l}$ are defined in the same way. By the Macchi-Soshnikov theorem, the operator $\Pi_{\mathcal{S}}^{q_1, \dots, q_l}$, a finite-rank perturbation of $\Pi_{\mathcal{S}}$, induces on the space of configurations on \mathbb{R} a determinantal process $\mathbb{P}_{\Pi_{\mathcal{S}}^{q_1, \dots, q_l}}$. Take distinct points

$$p_1, \dots, p_l, q_1, \dots, q_l \in \mathbb{R},$$

and, for a configuration X on \mathbb{R} write

$$\overline{\Psi}_N(p_1, \dots, p_l; q_1, \dots, q_l; X) = C_N \prod_{x \in X, |x| \leq N} \prod_{i=1}^l \left(\frac{x - p_i}{x - q_i} \right)^2,$$

where the constant C_N is chosen in such a way that

$$\int_{\text{Conf}(\mathbb{R})} \overline{\Psi}_N(p_1, \dots, p_l; q_1, \dots, q_l; X) d\mathbb{P}_{\Pi_{\mathcal{S}}^{q_1, \dots, q_l}} = 1.$$

Proposition 1.2. *The limit*

$$\overline{\Psi}(p_1, \dots, p_l; q_1, \dots, q_l; X) = \lim_{N \rightarrow \infty} \overline{\Psi}_N(p_1, \dots, p_l; q_1, \dots, q_l; X)$$

exists both $\mathbb{P}_{\mathcal{S}}$ -almost surely and $L_1(\text{Conf}(\mathbb{R}), \mathbb{P}_{\Pi_{\mathcal{S}}^{q_1, \dots, q_l}})$.

Proposition 1.3. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a diffeomorphism acting as the identity beyond a bounded open set $V \subset \mathbb{R}$. For $\mathbb{P}_{\mathcal{S}}$ -almost every configuration $X \in \text{Conf}(\mathbb{R})$ the following holds. If $X \cap V = \{q_1, \dots, q_l\}$, then*

$$\begin{aligned} (2) \quad \frac{d\mathbb{P}_{\mathcal{S}} \circ F}{d\mathbb{P}_{\mathcal{S}}}(X) &= \overline{\Psi}(F(q_1), \dots, F(q_l); q_1, \dots, q_l; X) \times \\ &\quad \times \frac{\det(\mathcal{S}(F(q_i), F(q_j)))_{i,j=1, \dots, l}}{\det(\mathcal{S}(q_i, q_j))_{i,j=1, \dots, l}} \times \\ &\quad \times F'(q_1) \dots F'(q_l). \end{aligned}$$

We now proceed to the formulation of the main result of this paper in full generality. We start by describing the assumptions on the kernels of our processes.

1.2. Integrable kernels. Let μ be a σ -finite Borel measure on \mathbb{R} ; for example, μ can be the Lebesgue measure on \mathbb{R} or on \mathbb{R}_+ or else the counting measure on \mathbb{Z} . The inner product in $L_2(\mathbb{R}, \mu)$ will be denoted \langle, \rangle . Let $L \subset L_2(\mathbb{R}, \mu)$ be a closed subspace, and let Π be the corresponding operator of orthogonal projection.

We assume that the operator Π is locally of trace class and admits a kernel, for which, slightly abusing notation, we keep the same symbol Π .

All kernels considered in this paper will always be supposed to satisfy the following

Assumption 1. *There exists a set $U \subset \mathbb{R}$, satisfying $\mu(\mathbb{R} \setminus U) = 0$, such that*

- (1) *For any $q \in U$ the function $v_q(x) = \Pi(x, q)$ lies in $L_2(\mathbb{R}, \mu)$ and for any $f \in L_2(\mathbb{R}, \mu)$ we have*

$$\Pi f(q) = \langle f, v_q \rangle.$$

In particular, all functions in L are defined everywhere on U .

- (2) *The diagonal values $\Pi(q, q)$ of the kernel Π are defined for all $q \in U$. We have $\langle v_q, v_q \rangle = \Pi(q, q)$, and, for any bounded Borel subset $B \subset \mathbb{R}$, we have*

$$\text{tr}(\chi_B \Pi \chi_B) = \int_B \Pi(q, q) d\mu(q).$$

- (3) *For any $q \in U$ and any $\varphi \in L$ satisfying $\varphi(q) = 0$, we have*

$$\frac{\varphi(x)}{x - q} \in L_2(\mathbb{R}, \mu).$$

The last condition is automatically satisfied once the kernel is sufficiently smooth: indeed, let $\varphi \in L$ have norm 1 and be such that $\varphi(q) = 0$, let

$$K^q(x, y) = K(x, y) - \frac{K(x, q)K(q, y)}{K(q, q)}$$

be the kernel of the orthogonal projection onto the space $L(q)$, the orthogonal complement of v_q in L . Finally, let \tilde{K} be the kernel of the orthogonal projection onto the orthogonal complement of φ in $L(q)$. For any $x \in U$, by definition, we have

$$K^q v_x = \langle v_x, \varphi \rangle v_x + \tilde{K} v_x,$$

whence, taking the inner product with v_x , we obtain

$$K^q(x, x) = |\varphi(x)|^2 + \langle \tilde{K} v_x, v_x \rangle.$$

In a small neighbourhood of q , we now have $K^q(x, x) = O(|x - q|^2)$, whence also $|\varphi(x)| = O(|x - q|)$, and the desired result follows.

We next assume that our kernel Π has integrable form : there exists an open set $U \subset \mathbb{R}$ satisfying $\mu(\mathbb{R} \setminus U) = 0$ and smooth functions A, B defined on U such that

$$(3) \quad \Pi(x, y) = \frac{A(x)B(y) - A(y)B(x)}{x - y}, x \neq y.$$

We assume that the functions A, B never simultaneously take value 0 on U . For $p \in U$ we have

$$v_p(x) = \frac{A(p)B(x) - B(p)A(x)}{p - x};$$

We have $v_p \in L_2(\mathbb{R}, \mu)$ for any $p \in U$ and for any $\varphi \in L_2(\mathbb{R}, \mu)$ we have

$$\Pi\varphi(p) = \langle \varphi, v_p \rangle.$$

We consider two cases:

- (1) the continuous case: for any $p \in \mathbb{R}$, $\mu(\{p\}) = 0$;
- (2) the discrete case: μ is the counting measure on a countable subset $E \subset \mathbb{R}$ without accumulation points.

In the continuous case we make the additional requirement

$$(4) \quad \Pi(x, x) = A'(x)B(x) - A(x)B'(x).$$

on diagonal values of the kernel Π ; in the discrete case, when the measure μ is the counting measure on a countable subset $E \subset \mathbb{R}$ without accumulation points, the integrability assumption only concerns off-diagonal entries of the kernel $\Pi(x, y)$, and the smoothness assumption is not needed: A, B are just arbitrary functions defined on E . Note also that the third requirement of Assumption 1 is only needed in the continuous case.

As before, given l distinct points $q_1, \dots, q_l \in \mathbb{R}$, we set

$$L(q_1, \dots, q_l) = \{\varphi \in L : \varphi(q_1) = \dots = \varphi(q_l) = 0\};$$

and we denote Π^{q_1, \dots, q_l} the operator of orthogonal projection onto the subspace $L(q_1, \dots, q_l)$.

Remark. The functions A, B in the definition of integrability are not unique: for example, if one makes a linear unimodular change of variable

$$(5) \quad (A, B) \rightarrow (\alpha_{11}A + \alpha_{12}B, \alpha_{21}A + \alpha_{22}B), \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1,$$

then the formula (3) remains valid.

1.3. The main result in the continuous case. In this subsection we assume that the measure μ is continuous (for any $p \in \mathbb{R}$, $\mu(\{p\}) = 0$) and additionally that

$$(6) \quad \int_{\mathbb{R}} \frac{\Pi(x, x)}{1 + x^2} d\mu(x) < +\infty.$$

Let $p_1, \dots, p_l, q_1, \dots, q_l \in \mathbb{R}$ be distinct. For $R > 0$, $\varepsilon > 0$ and a configuration X on \mathbb{R} write

$$\overline{\Psi}_{R,\varepsilon}(p_1, \dots, p_l; q_1, \dots, q_l; X) = C(R, \varepsilon) \times \prod_{x \in X, |x| \leq R, \min_{i=1, \dots, l} |x - q_i| \geq \varepsilon} \prod_{i=1}^l \left(\frac{x - p_i}{x - q_i} \right)^2,$$

where the constant $C(R, \varepsilon)$ is chosen in such a way that

$$(7) \quad \int_{\text{Conf}(\mathbb{R})} \overline{\Psi}_{R,\varepsilon}(p_1, \dots, p_l; q_1, \dots, q_l; X) d\mathbb{P}_{\Pi}^{q_1, \dots, q_l} = 1.$$

Proposition 1.4. *If the kernel Π of an orthogonal projection operator, for which Assumption 1 holds, is integrable and satisfies (6), then the limit*

$$\overline{\Psi}(p_1, \dots, p_l; q_1, \dots, q_l; X) = \lim_{R \rightarrow \infty, \varepsilon \rightarrow 0} \overline{\Psi}_{R,\varepsilon}(p_1, \dots, p_l; q_1, \dots, q_l; X)$$

exists both $\mathbb{P}_{\Pi^{q_1, \dots, q_l}}$ -almost surely and in $L_1(\text{Conf}(\mathbb{R}), \mathbb{P}_{\Pi^{q_1, \dots, q_l}})$.

Theorem 1.5. *Let μ be a continuous measure on \mathbb{R} . Let Π be an integrable kernel satisfying Assumption 1 as well as (6) and inducing a locally trace-class operator of orthogonal projection in $L_2(\mathbb{R}, \mu)$. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel automorphism preserving the measure class of μ and acting as the identity beyond a bounded open set $V \subset \mathbb{R}$. For \mathbb{P}_{Π} -almost every configuration $X \in \text{Conf}(\mathbb{R})$ the following holds. If $X \cap V = \{q_1, \dots, q_l\}$, then*

$$(8) \quad \frac{d\mathbb{P}_{\Pi} \circ F}{d\mathbb{P}_{\Pi}}(X) = \overline{\Psi}(F(q_1), \dots, F(q_l); q_1, \dots, q_l; X) \times \frac{\det(\Pi(F(q_i), F(q_j)))_{i,j=1, \dots, l}}{\det(\Pi(q_i, q_j))_{i,j=1, \dots, l}} \times \frac{d\mu \circ F}{d\mu}(q_1) \dots \frac{d\mu \circ F}{d\mu}(q_l).$$

Remark. The open set V can be chosen in many ways; the resulting value of the Radon-Nikodym derivative is of course the same.

For example, Theorem 1.5 applies to the sine-process as well as to the Airy and Bessel point processes of Tracy and Widom [26], [27].

1.4. The main result in the discrete case. The main result is similar in the discrete case except that we also need to consider measures conditional on the absence of particles and that, in order to ensure quasi-invariance of our measures under the infinite symmetric group, we impose the extra restriction that our subspace L not contain functions with finite support.

Let $E \subset \mathbb{R}$ be a countable subset without accumulation points, endowed with the counting measure. In what follows, we will need the assumption

$$(9) \quad \sum_{n \in E} \frac{1}{1+n^2} < +\infty.$$

Let Π be an integrable kernel inducing an operator of orthogonal projection onto a subspace $L \subset L_2(E)$, and let \mathbb{P}_Π be the corresponding determinantal measure on the space $\text{Conf}(E)$ of configurations on E . The infinite symmetric group naturally acts on E by finite permutations and induces the corresponding natural action on $\text{Conf}(E)$. Given $l \in \mathbb{N}$, $m < l$ and an l -tuple (p_1, \dots, p_l) of distinct points in E such that there does not exist a nonzero function in L supported on the set $\{p_1, \dots, p_l\}$, we introduce a closed subspace $L(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$ by the formula

$$(10) \quad L(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l) = \{ \chi_{E \setminus \{p_{m+1}, \dots, p_l\}} \varphi : \varphi \in L, \varphi(p_1) = \dots = \varphi(p_l) = 0 \}.$$

Let $\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}$ be the corresponding orthogonal projection operator, $\mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}}$ the corresponding determinantal measure.

Take $R > 0$, $m \leq l$, a permutation σ of the points p_1, \dots, p_l , and define

$$\overline{\Psi}_R(p_1, \dots, p_l, m, \sigma; X) = C_R \prod_{x \in X: |x| \leq R} \prod_{i=1}^m \left(\frac{x - \sigma(p_i)}{x - p_i} \right)^2 \chi_{E \setminus \{p_1, \dots, p_l\}}(x),$$

where the positive constant C_R is chosen in such a way that

$$\int_{\text{Conf}(E)} \overline{\Psi}_R(p_1, \dots, p_l, m, \sigma) d\mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}} = 1.$$

Remark. Our multiplicative functional is positive with positive probability precisely because the subspace L does not admit functions supported on $\{p_1, \dots, p_l\}$.

Proposition 1.6. *Let E be a countable subset of \mathbb{R} without accumulation points satisfying (9). Let Π be an integrable kernel inducing an operator of orthogonal projection on $L_2(E)$. Let $p_1, \dots, p_l \in E$ be distinct points such that there does not exist a nonzero function in L supported on the set $\{p_1, \dots, p_l\}$. Then, for any $m \leq l$ and any permutation σ of p_1, \dots, p_l , the*

limit

$$\overline{\Psi}(p_1, \dots, p_l, m, \sigma) = \lim_{R \rightarrow \infty} \overline{\Psi}_R(p_1, \dots, p_l, m, \sigma)$$

exists $\mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}}$ -almost surely and in $L_1(\text{Conf}(E), \mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}})$.

Let

$$\mathbf{C}(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$$

be the set of configurations on E containing exactly one particle in each of the positions p_1, \dots, p_m and no particles in the positions p_{m+1}, \dots, p_l .

We are now ready to formulate the main result in the discrete case, the quasi-invariance of determinantal measures with integrable kernels under the natural action of the infinite symmetric group on $\text{Conf}(E)$. Given a permutation σ of points p_1, \dots, p_l of the set E , slightly abusing notation, we use the same symbol both for the bijection of E that acts as σ on $\{p_1, \dots, p_l\}$ and as the identity on $E \setminus \{p_1, \dots, p_l\}$ and the automorphism induced by this bijection on the space $\text{Conf}(E)$ of configurations on E .

Theorem 1.7. *Let E be a countable subset of \mathbb{R} without accumulation points satisfying (9). Let Π be an integrable kernel inducing an operator of orthogonal projection onto a closed subspace $L \subset L_2(E)$. Let p_1, \dots, p_l be distinct points in E such that there does not exist a nonzero function in L supported on the set $\{p_1, \dots, p_l\}$. Then for any $m \leq l$, any permutation σ of the points p_1, \dots, p_l and \mathbb{P}_{Π} -almost every $X \in \mathbf{C}(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$, we have*

(11)

$$\frac{d\mathbb{P}_{\Pi} \circ \sigma}{d\mathbb{P}_{\Pi}}(X) = \overline{\Psi}(p_1, \dots, p_l, m, \sigma; X) \times \frac{\det(\Pi(\sigma(p_i), \sigma(p_j)))_{i,j=1, \dots, m}}{\det(\Pi(p_i, p_j))_{i,j=1, \dots, m}}.$$

In particular, if the subspace L does not contain functions supported on finite sets, then the measure \mathbb{P}_{Π} is quasi-invariant under the natural action of the infinite symmetric group on $\text{Conf}(E)$.

For example, the discrete sine-process of Borodin, Okounkov and Olshanski [2] as well as the Gamma kernel process of Borodin and Olshanski [3] satisfy all the assumptions of Theorem 1.7.

Remark. By the Theorem of Ghosh [8], the sine-process, discrete or continuous, is *rigid*: if, for a bounded subset B and a configuration X , we let $\#_B(X)$ stand for the number of particles of X lying in B , and, for any Borel subset C we let \mathcal{F}_C be the σ -algebra generated by all random variables of the form $\#_B, B \subset C$, then, for any bounded B , the random variable $\#_B$ is measurable with respect to the completion, under the sine-process, of the sigma-algebra \mathcal{F}_{B^c} , where B^c stands for the complement of B . As R. Lyons, developing the method of [1], showed in Theorem 7.15 of [15], the tail sigma-algebra of the discrete sine-process is trivial. It follows now

that the *symmetric* sigma-algebra of the sine-process is trivial as well: in other words, the discrete sine-process is ergodic with respect to the action of the infinite symmetric group. This argument holds, of course, for any rigid point process.

To further illustrate Theorem 1.7, we now write the Radon-Nikodym derivative for a transposition of two points $p, q \in E$. Set

$$L(p, \check{q}) = \{\chi_{\mathbb{Z} \setminus \{p, q\}} \varphi, \varphi \in L, \varphi(p) = 0\}.$$

and let $\mathbb{P}_{\Pi}^{p, \check{q}}$ be the determinantal measure corresponding to the operator of orthogonal projection onto the subspace $L(p, \check{q})$. The subspace $L(q, \check{p})$ and the measure $\mathbb{P}_{\Pi}^{q, \check{p}}$ are defined in the same way. Write

$$\overline{\Psi}_N(p, q; X) = C_{p, q} \times \prod_{x \in X, |x| \leq N} \left(\frac{x - p}{x - q} \right)^2,$$

where the constant $C_{p, q}$ is chosen in such a way that

$$\int_{\text{Conf}(E)} \overline{\Psi}_N(p, q; X) d\mathbb{P}_{\Pi}^{p, \check{q}}(X) = 1.$$

By definition, $\mathbb{P}_{\Pi}^{p, \check{q}}$ -almost all configurations X on E contain no particles either at p or at q , so the function $\overline{\Psi}_N$ is well-defined; by definition it is bounded.

Proposition 1.8. *The limit $\overline{\Psi}(p, q; X) = \lim_{N \rightarrow \infty} \overline{\Psi}_N(p, q; X)$ exists both $\mathbb{P}_{\Pi}^{p, \check{q}}$ -almost surely and in $L_1(\text{Conf}(E), \mathbb{P}_{\Pi}^{p, \check{q}})$.*

The regularized multiplicative functional $\overline{\Psi}(q, p; X)$ is defined in the same way.

The Radon-Nikodym derivative of \mathbb{P}_{Π} under the action of the permutation σ_{pq} is now given by the following

Proposition 1.9. *For \mathbb{P}_{Π} -almost all $X \in \text{Conf}(E)$ the following holds.*

If $p \notin X, q \in X$, then

$$\frac{d\mathbb{P}_{\Pi} \circ \sigma_{pq}}{d\mathbb{P}_{\Pi}}(X) = \overline{\Psi}(p, q; X) \cdot \frac{\Pi(p, p)}{\Pi(q, q)}.$$

If $p \in X, q \notin X$, then

$$\frac{d\mathbb{P}_{\Pi} \circ \sigma_{pq}}{d\mathbb{P}_{\Pi}}(X) = \overline{\Psi}(q, p; X) \cdot \frac{\Pi(q, q)}{\Pi(p, p)}.$$

If $p, q \in X$ or $p, q \notin X$, then

$$\frac{d\mathbb{P}_{\Pi} \circ \sigma_{pq}}{d\mathbb{P}_{\Pi}}(X) = 1.$$

Remark. If E is a countable set, \mathbb{P} a Gibbs measure on $\text{Conf}(E)$ corresponding to the Hamiltonian H of pairwise interaction of particles (cf. e.g. Sinai [24]), p, q are points in E and σ_{pq} the transposition of p and q , then, for almost every configuration X , conditioned to contain a particle at q but not at p , by definition, we have

$$\frac{d\mathbb{P} \circ \sigma_{p,q}}{d\mathbb{P}}(X) = \prod_{x \in X: x \neq q} \exp(H(p, x) - H(q, x)).$$

The quasi-invariance property established in this paper is thus, informally speaking, the analogue of the Gibbs property, with Hamiltonian $H(x, y) = 2 \log |x - y|$, for determinantal point processes.

1.5. Examples of regularized multiplicative functionals. Regularization of a multiplicative functional can take different form depending on the specific process. We illustrate this by two examples.

The Sine-Process. The argument below is valid for the continuous sine-process as well as the discrete sine-process. The sine-process is stationary, therefore, for almost every configuration X the series

$$(12) \quad \sum_{x \in X: x \neq 0} \frac{1}{x}$$

diverges absolutely since so does the harmonic series. Nonetheless, the series (12) converges *conditionally* in principal value: the limit

$$\lim_{N \rightarrow \infty} \sum_{x \in X: x \neq 0, |x| \leq N} \frac{1}{x}$$

is almost surely finite and, as we shall check below, has finite variance. Similarly, for distinct points $p_1, \dots, p_l, q_1, \dots, q_l$, taken in \mathbb{R} in the continuous case and in \mathbb{Z} in the discrete case,

$$(13) \quad \lim_{N \rightarrow \infty} \prod_{x \in X, |x| \leq N, x \neq q_1, \dots, q_l} \prod_{i=1}^l \left(\frac{x - p_i}{x - q_i} \right)^2,$$

the limit exists and has finite expectation. The normalized multiplicative functional is in this case precisely the limit (13) normalized to have expectation 1.

The Determinantal Point Process with the Gamma-Kernel. The determinantal point process with the Gamma-kernel, introduced by Borodin and Olshanski in [3] and for which the quasi-invariance under the action of the infinite symmetric group is due to Olshanski [18], is a point process on the phase space $\mathbb{Z}' = 1/2 + \mathbb{Z}$ of half-integers such that for almost every

configuration X we have

$$(14) \quad \sum_{x \in X: x > 0} \frac{1}{x} < +\infty, \quad \sum_{y \notin X: y < 0} \frac{1}{|y|} < +\infty.$$

Furthermore, each sum in (14), considered as a random variable on the space of configurations on \mathbb{Z}' , has finite variance with respect to the determinantal point process with the Gamma-kernel.

For $p, q \in \mathbb{Z}'$, the normalized multiplicative functional corresponding to the function $g^{p,q}(x) = ((x-p)/(x-q))^2$ will therefore have the form

$$C \cdot \prod_{x \in X, x > 0} g(x) \cdot \prod_{y \notin X: y < 0} g^{-1}(y),$$

where the constant C is chosen in such a way that the resulting expression have expectation 1.

1.6. Outline of the argument. We start with the discrete case and illustrate the argument in the specific case of a transposition of two distinct points $p, q \in E$. A theorem due to Lyons [15], Shirai-Takahashi [22] states that the measure $\mathbb{P}_{\Pi}^{p,\check{q}}$ is the conditional measure of \mathbb{P}_{Π} on the subset of configurations containing a particle at p and not containing a particle at q .

Step 1. The Relation Between Palm Subspaces. The key point in the proof of Proposition 1.9 is the equality

$$(15) \quad L(p, \check{q}) = \frac{x-p}{x-q} L(q, \check{p}),$$

which it is more convenient to rewrite in the form

$$(16) \quad L(p, \check{q}) = \chi_{E \setminus \{p, q\}} \frac{x-p}{x-q} L(q, \check{p}).$$

The equality (16) directly follows from the *integrability* of the discrete sine-kernel. The remainder of the argument shows that the relation (16) implies the relation

$$(17) \quad \mathbb{P}_{\Pi}^{p,\check{q}} = \overline{\Psi}(p, q) \mathbb{P}_{\Pi}^{q,\check{p}},$$

which, in turn, is a reformulation of Proposition 1.9.

Step 2. Multiplicative functionals of determinantal point processes. Given a function g on \mathbb{Z} , the multiplicative functional Ψ_g is defined on $\text{Conf}(E)$ by the formula

$$\Psi_g(X) = \prod_{x \in X} g(x).$$

provided that the infinite product in the right-hand side converges absolutely.

At the centre of the argument lies the result of [6] that can informally be summarized as follows: a determinantal measure times a multiplicative

functional is, after normalization, again a determinantal measure. More precisely, let g be a positive function on E bounded away from 0 and ∞ , and let Π be an operator of orthogonal projection in $L_2(E)$ onto a closed subspace L . Let Π^g be the operator of orthogonal projection onto the subspace $\sqrt{g}L$. Then, under certain additional assumptions we have

$$(18) \quad \mathbb{P}_{\Pi^g} = \frac{\Psi_g \mathbb{P}_{\Pi}}{\int_{\text{Conf}(E)} \Psi_g d\mathbb{P}_{\Pi}}$$

The relation (18) together with the relation (16) suggests that the measures $\mathbb{P}_{\Pi}^{p,\check{q}}$ and $\mathbb{P}_{\Pi}^{q,\check{p}}$ are equivalent, and the Radon-Nikodym derivative is given by the normalized multiplicative functional corresponding to the function

$$g^{p,\check{q}}(x) = \frac{x-p}{x-q} \chi_{E \setminus \{p,q\}}.$$

Step 3. Regularization of multiplicative functionals. A technical difficulty arises that in many examples the multiplicative functional corresponding to the function $g^{p,q}$ might fail to converge absolutely with respect to the measure $\mathbb{P}_{\Pi}^{p,\check{q}}$; indeed, in many examples (in particular, for stationary determinantal processes on \mathbb{Z}), we have

$$\sum_{x \in E} |g^{p,q}(x) - 1| = +\infty$$

and, consequently, also

$$\sum_{x \in E} |g^{p,q}(x) - 1| \cdot \Pi^{p,\check{q}}(x, x) = +\infty.$$

In order to resolve this difficulty, we go back to the formula (18). For multiplicative functional Ψ_g integrable with respect to a determinantal measure \mathbb{P}_{Π} set

$$(19) \quad \overline{\Psi}_g = \frac{\Psi_g}{\int \Psi_g d\mathbb{P}_{\Pi}}.$$

The functional $\overline{\Psi}_g$ will be called the normalized multiplicative functional corresponding to Ψ_g and \mathbb{P}_{Π} . To keep notation lighter, we do not explicitly indicate dependence on Π ; in what follows, the precise measure, with respect to which normalization is taken, will be clear from the context. We now rewrite (18) in the form

$$(20) \quad \mathbb{P}_{\Pi^g} = \overline{\Psi}_g \cdot \mathbb{P}_{\Pi}.$$

The key observation for the remainder of the argument is that the definition of the normalized multiplicative functional $\overline{\Psi}_g$ can be extended in such a way that (20) continues to hold for a wider class of functions g , for which the multiplicative functional itself diverges almost surely.

We first explain the idea of this extension for additive functionals. Given a measurable function f on E , the corresponding additive functional on $\text{Conf}(E)$ is defined by the formula

$$S_f(X) = \sum_{x \in X} f(x)$$

provided the series in the right hand side converges absolutely. The expectation of the additive functional with respect to \mathbb{P}_Π is given by the formula

$$(21) \quad \mathbb{E}_{\mathbb{P}_\Pi} S_f = \sum_{x \in E} f(x) \Pi(x, x),$$

provided, again, that the series in the right hand side converges absolutely. For the variance of the additive functional we have

$$(22) \quad \text{Var}_{\mathbb{P}_\Pi} S_f = \sum_{x, y \in \mathbb{Z}} (f(x) - f(y))^2 (\Pi(x, y))^2.$$

Let

$$\overline{S}_f = S_f - \mathbb{E}_{\mathbb{P}_\Pi} S_f$$

be the normalized additive functional corresponding to the function f . It is easy to give examples of functions f for which the sum in the right hand side of (21) diverges while the sum in the right hand side of (22) converges. For such functions, convergence of the sum in the right hand side of (22) allows one to define the normalized additive functional \overline{S}_f by continuity, even though the additive functional S_f itself is not defined. In a similar way, for a function g bounded away from 0 and ∞ and satisfying

$$\sum_{x \in E} |g(x) - 1|^2 \Pi(x, x) < +\infty,$$

one can define the normalized multiplicative functional $\overline{\Psi}_g$, even though the multiplicative functional Ψ_g itself need not be defined; the relation (20) still holds.

We thus check that the normalized multiplicative functional $\overline{\Psi}_{g^{p,q}}$ can be defined with respect to the measure $\mathbb{P}_\Pi^{q,\vec{p}}$; the relation (16) now implies the desired equality (17). This completes the outline of the proof of Theorem 1.7.

The proof in continuous case follows a similar scheme. The rôle of conditional measures is played by reduced Palm measures. The reduced

Palm measure $\mathbb{P}_{\Pi}^{q_1, \dots, q_l}$ of the measure \mathbb{P}_{Π} with respect to l distinct points $q_1, \dots, q_l \in \mathbb{R}$ is the determinantal measure corresponding to the operator Π^{q_1, \dots, q_l} of the orthogonal projection onto the subspace

$$L(q_1, \dots, q_l) = \{\varphi \in L_{\mathcal{S}} : \varphi(q_1) = \dots = \varphi(q_l) = 0\}.$$

The continuous analogue of the equality (16) is the relation

$$(23) \quad L(p_1, \dots, p_l) = \frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)} L(q_1, \dots, q_l)$$

valid for μ -almost any two l -tuples of distinct points $(p_1, \dots, p_l), (q_1, \dots, q_l)$ in \mathbb{R} .

The next step is to regularize the multiplicative functional corresponding to the function

$$(24) \quad \frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)},$$

while the overall scheme of regularization is the same as in the discrete case, additional estimates are needed here because the function (24) is bounded away neither from zero nor from infinity.

The resulting normalized multiplicative functional $\overline{\Psi}(p_1, \dots, p_l, q_1, \dots, q_l)$ is then seen to be the Radon-Nikodym derivative of the reduced Palm measures $\mathbb{P}_{\Pi}^{p_1, \dots, p_l}$ and $\mathbb{P}_{\Pi}^{q_1, \dots, q_l}$, which, in turn, implies Theorem 1.5.

1.7. Organization of the paper. The paper is organized as follows. In Section 2, we collect necessary facts about determinantal point processes, their multiplicative functionals and their Palm measures. We recall the results of [6] (see also [5]) showing that the product of a determinantal measure with a multiplicative functional is, after normalization, again a determinantal measure, whose kernel is found explicitly. We also check that equivalence of reduced Palm measures corresponding to distinct l -tuples of points implies the quasi-invariance of the point process under Borel automorphisms preserving the class of its correlation measures and acting by the identity beyond a bounded set. In Section 3, we start by showing that reduced Palm measures of determinantal point processes given by projection operators with integrable kernels are themselves determinantal point processes given by projection operators with integrable kernels and proceed to verify the key relations (50) and (54) showing that the ranges of projection operators corresponding to reduced Palm measures at distinct points differ by multiplication by a function.

Proposition 4.6 in Section 4 describes the regularization of multiplicative functionals. Relations (50) and (54) are then seen to imply that the

reduced Palm measures themselves are equivalent, and that the corresponding Radon-Nikodym derivative is a regularized multiplicative functional. This completes the proof of Theorems 1.5 and 1.7.

Acknowledgements. Grigori Olshanski posed the problem to me and suggested that the Radon-Nikodym derivative be given by a multiplicative functional; I am greatly indebted to him. I am deeply grateful to Alexei Klimenko and Cosme Louart for useful discussions.

This work has been carried out thanks to the support of the A*MIDEX project (no. ANR-11-IDEX-0001-02) funded by the programme “Investissements d’Avenir” of the Government of the French Republic, managed by the French National Research Agency (ANR). It was also supported in part by the Grant MD-2859.2014.1 of the President of the Russian Federation, by the Programme “Dynamical systems and mathematical control theory” of the Presidium of the Russian Academy of Sciences, by the ANR under the project “VALET” (ANR-13-JS01-0010) of the Programme JCJC SIMI 1, and by the RFBR grant 13-01-12449.

Part of this work was done while I was visiting Institut Henri Poincaré in Paris and the Max Planck Institute in Bonn; I am deeply grateful to these institutions for their warm hospitality.

Remark. After this paper was completed, I became aware of the preprint *Absolute continuity and singularity of Palm measures of the Ginibre point process*, arXiv:1406.3913, by Hirofumi Osada and Tomoyuki Shirai, in which, for the Ginibre ensemble, using its finite-dimensional approximations by orthogonal polynomial ensembles, the authors establish the equivalence of reduced Palm measures, conditioned at distinct l -tuples of points in \mathbb{C} , and represent the Radon-Nikodym derivative as a regularized multiplicative functional.

2. POINT PROCESSES AND PALM DISTRIBUTIONS.

2.1. Spaces of configurations. Let E be a locally compact complete metric space. A *configuration* on E is a collection of points in E , called *particles*, considered without regard to order and subject to the additional requirement that every bounded set contain only finitely many particles of a configuration. Let $\text{Conf}(E)$ be the space of configurations on E . To a configuration $X \in \text{Conf}(E)$ assign a Radon measure

$$\sum_{x \in X} \delta_x$$

on the space E ; this correspondence identifies the space $\text{Conf}(E)$ with the space of integer-valued Radon measures on E . The space $\text{Conf}(E)$ is thus endowed with a natural structure of a complete separable metric space. The

Borel structure on the space $\text{Conf}(E)$ can equivalently be defined without introducing a topology: namely, for a bounded Borel set $B \subset E$, let

$$\#_B : \text{Conf}(E) \rightarrow \mathbb{N} \cup \{0\}$$

be the function that to a configuration assigns the number of its particles belonging to B . The random variables $\#_B$ over all bounded Borel sets $B \subset E$ determine the Borel sigma-algebra on $\text{Conf}(E)$.

2.2. Multiplicative functionals. We next recall the definition of *multiplicative functionals* on spaces of configurations. Let g be a non-negative measurable function on E , and introduce the *multiplicative functional* $\Psi_g : \text{Conf}(E) \rightarrow \mathbb{R}$ by the formula

$$(25) \quad \Psi_g(X) = \prod_{x \in X} g(x).$$

If the infinite product $\prod_{x \in X} g(x)$ absolutely converges to 0 or to ∞ , then we set, respectively, $\Psi_g(X) = 0$ or $\Psi_g(X) = \infty$. If the product in the right-hand side fails to converge absolutely, then the multiplicative functional is not defined.

2.3. Point processes. A Borel probability measure \mathbb{P} on $\text{Conf}(E)$ is called a *point process* with phase space E .

We recall that the process \mathbb{P} is said to admit correlation functions of order l if for any continuous compactly supported function f on E^l the functional

$$\sum_{x_1, \dots, x_l \in X} f(x_1, \dots, x_l)$$

is \mathbb{P} -integrable; here the sum is taken over all l -tuples of distinct particles in X . The l -th correlation measure ρ_l of the point process \mathbb{P} is then defined by the formula

$$\mathbb{E}_{\mathbb{P}} \left(\sum_{x_1, \dots, x_l \in X} f(x_1, \dots, x_l) \right) = \int_{E^l} f(q_1, \dots, q_l) d\rho_l(q_1, \dots, q_l).$$

By definition, a point process \mathbb{P} is uniquely determined by prescribing joint distributions, with respect to \mathbb{P} , of random variables $\#_{B_1}, \dots, \#_{B_l}$ over all finite collections of disjoint bounded Borel subsets $B_1, \dots, B_l \subset E$. Since, for arbitrary nonzero complex numbers z_1, \dots, z_l inside the unit circle, the function

$$(26) \quad \prod_{k=1}^l z_k^{\#_{B_k}}$$

is a well-defined multiplicative functional on $\text{Conf}(E)$, that, moreover, takes values inside the unit circle, a point process \mathbb{P} on $\text{Conf}(E)$ is also uniquely determined by prescribing the values of expectations of multiplicative functionals of the form (26).

2.4. Campbell Measures. Following Kallenberg [12] and Daley–Vere-Jones [7], we now recall the definition of Campbell measures of point processes.

Take a Borel probability measure \mathbb{P} on $\text{Conf}(E)$ of *finite local intensity*, that is, admitting the first correlation measure ρ_1 , or, equivalently, such that for any bounded Borel set B , the function $\#_B$ is integrable with respect to \mathbb{P} . For any bounded Borel set $B \subset E$, by definition we then have

$$\rho_1(B) = \int_{\text{Conf}(E)} \#_B(X) d\mathbb{P}(X).$$

The *Campbell measure* $\mathcal{C}_{\mathbb{P}}$ of a Borel probability measure \mathbb{P} of finite local intensity on $\text{Conf}(E)$ is a sigma-finite measure on $E \times \text{Conf}(E)$ such that for any Borel subsets $B \subset E$, $\mathcal{Z} \subset \text{Conf}(E)$ we have

$$\mathcal{C}_{\mathbb{P}}(B \times \mathcal{Z}) = \int_{\mathcal{Z}} \#_B(X) d\mathbb{P}(X).$$

For a point process admitting correlation functions of order l one can also define the l -th iterated Campbell measure $\mathcal{C}^{(l)}$ of the point process \mathbb{P} , that is, by definition, a measure on $E^l \times \text{Conf}(E)$ such that for any disjoint bounded sets $B_1, \dots, B_l \subset E$ and any measurable subset $\mathcal{Z} \subset \text{Conf}(E)$ we have

$$(27) \quad \mathcal{C}^{(l)}(B_1 \times \dots \times B_l \times \mathcal{Z}) = \int_{\mathcal{Z}} \#_{B_1}(X) \times \dots \times \#_{B_l}(X) d\mathbb{P}(X)$$

2.5. Palm Distributions. Following Kallenberg [12] and Daley–Vere-Jones [7], we now recall the construction of Palm distributions from Campbell measures. For a fixed Borel $\mathcal{Z} \subset \text{Conf}(E)$ the Campbell measure $\mathcal{C}_{\mathbb{P}}$ induces a sigma-finite measure $\mathcal{C}_{\mathbb{P}}^{\mathcal{Z}}$ on E by the formula

$$\mathcal{C}_{\mathbb{P}}^{\mathcal{Z}}(B) = \mathcal{C}_{\mathbb{P}}(B \times \mathcal{Z}).$$

By definition, for any Borel subset $\mathcal{Z} \subset \text{Conf}(E)$ the measure $\mathcal{C}_{\mathbb{P}}^{\mathcal{Z}}$ is absolutely continuous with respect to ρ_1 . For ρ_1 -almost every $q \in E$, one can therefore define a probability measure $\hat{\mathbb{P}}^q$ on $\text{Conf}(E)$ by the formula

$$\hat{\mathbb{P}}^q(\mathcal{Z}) = \frac{d\mathcal{C}_{\mathbb{P}}^{\mathcal{Z}}}{d\rho_1}(q).$$

The measure $\hat{\mathbb{P}}^q$ is called *the Palm measure* of \mathbb{P} at the point q . Equivalently, the Palm measure $\hat{\mathbb{P}}^q$ is the canonical conditional measure, in the sense of

Rohlin [20], of the Campbell measure $\mathcal{C}_{\mathbb{P}}$ with respect to the measurable partition of the space $E \times \text{Conf}(E)$ into subsets of the form $\{q\} \times \text{Conf}(E)$, $q \in E$.

Similarly, using iterated Campbell measures one defines iterated Palm measures: for a fixed Borel $\mathcal{Z} \subset \text{Conf}(E)$ the l -th iterated Campbell measure $\mathcal{C}_{\mathbb{P}}^l$ induces a sigma-finite measure $\mathcal{C}_{\mathbb{P}}^{l,\mathcal{Z}}$ on E by the formula

$$\mathcal{C}_{\mathbb{P}}^{l,\mathcal{Z}}(B) = \mathcal{C}_{\mathbb{P}}(B \times \mathcal{Z}).$$

By definition, for any Borel subset $\mathcal{Z} \subset \text{Conf}(E)$ the measure $\mathcal{C}_{\mathbb{P}}^{l,\mathcal{Z}}$ is absolutely continuous with respect to ρ_l . For ρ_l -almost all $(q_1, \dots, q_l) \in E^l$, one can therefore define a probability measure $\hat{\mathbb{P}}^{q_1, \dots, q_l}$ on $\text{Conf}(E)$ by the formula

$$\hat{\mathbb{P}}^{q_1, \dots, q_l}(\mathcal{Z}) = \frac{d\mathcal{C}_{\mathbb{P}}^{\mathcal{Z}}}{d\rho_l}(q_1, \dots, q_l).$$

The measure $\hat{\mathbb{P}}^{q_1, \dots, q_l}$ is called the l -th iterated Palm measure of the point process \mathbb{P} . The iterated Palm measure $\hat{\mathbb{P}}^q$ is the canonical conditional measure, in the sense of Rohlin [20], of the Campbell measure $\mathcal{C}_{\mathbb{P}}^l$ with respect to the measurable partition of the space $E^l \times \text{Conf}(E)$ into subsets of the form $\{q_1, \dots, q_l\} \times \text{Conf}(E)$, with $q_1, \dots, q_l \in E$ distinct.

For distinct points q_1, \dots, q_l , the l -th iterated Palm measure of course satisfies

$$\hat{\mathbb{P}}^{q_1, \dots, q_l} = \left(\dots \left(\hat{\mathbb{P}}^{q_1} \right)^{q_2} \dots \right)^{q_l}.$$

2.6. Reduced Palm measures. By definition, the Palm measure $\hat{\mathbb{P}}^{q_1, \dots, q_l}$ is supported on the subset of configurations containing a particle at each position q_1, \dots, q_l . It is often convenient to remove these particles and to define the *reduced* Palm measure $\mathbb{P}^{q_1, \dots, q_l}$ as the push-forward of the Palm measure $\hat{\mathbb{P}}^{q_1, \dots, q_l}$ under the erasing map $X \rightarrow X \setminus \{q_1, \dots, q_l\}$. Reduced Palm measures allow one to give a convenient representation for measures of cylinder sets. Take $X_0 \in \text{Conf}(E)$ and $q_1^{(0)}, \dots, q_l^{(0)} \in X_0$. Take disjoint bounded open sets $B^{(1)}, \dots, B^{(l)} \subset E$, set $B = \cup B^{(i)}$ and take an open set $U \subset E$ disjoint from all $B^{(i)}$ in such a way that $q_i^{(0)} \in B^{(i)}$ for all $i = 1, \dots, l$ and $X_0 \setminus \{q_1^{(0)}, \dots, q_l^{(0)}\} \subset U$. Let \mathcal{W} be a neighbourhood of $X_0 \setminus \{q_1^{(0)}, \dots, q_l^{(0)}\}$ in $\text{Conf}(E)$ satisfying

$$\mathcal{W} \subset \{X \in \text{Conf}(E) : X \subset U\}.$$

Introduce a neighbourhood \mathcal{Z} of X_0 by setting

(28)

$$\mathcal{Z} = \{X \in \text{Conf}(E) : \#_{B^{(1)}}(X) = \dots = \#_{B^{(l)}}(X) = 1, X|_{E \setminus B} \subset \mathcal{W}\}.$$

Proposition 2.1. *We have*

$$\mathbb{P}(\mathcal{Z}) = \int_{B^{(1)} \times \dots \times B^{(l)}} \mathbb{P}^{q_1, \dots, q_l}(\mathcal{Z}) d\rho_l(q_1, \dots, q_l).$$

Proof. By definition of iterated Palm measures, we have

$$\mathcal{C}^{(l)}(B^{(1)} \times \dots \times B^{(l)} \times \mathcal{Z}) = \int_{B^{(1)} \times \dots \times B^{(l)}} \hat{\mathbb{P}}^{q_1, \dots, q_l}(\mathcal{Z}) d\rho_l(q_1, \dots, q_l).$$

By construction and definition of reduced Palm measures, we have

$$\hat{\mathbb{P}}^{q_1, \dots, q_l}(\mathcal{Z}) = \mathbb{P}^{q_1, \dots, q_l}(\mathcal{W}).$$

By definition, see (27),(28), we have

$$\mathcal{C}^{(l)}(B^{(1)} \times \dots \times B^{(l)} \times \mathcal{Z}) = \int_{\mathcal{Z}} \#_{B^{(1)}}(X) \times \dots \times \#_{B^{(l)}}(X) d\mathbb{P}(\mathcal{Z}) = \mathbb{P}(\mathcal{Z}).$$

Consequently,

$$\mathbb{P}(\mathcal{Z}) = \int_{B^{(1)} \times \dots \times B^{(l)}} \mathbb{P}^{q_1, \dots, q_l}(\mathcal{W}) d\rho_l(q_1, \dots, q_l),$$

as desired.

2.7. Locally trace class operators and their kernels. Let μ be a sigma-finite Borel measure on E . The inner product in $L_2(E, \mu)$ is always denoted by the symbol \langle, \rangle .

Let $\mathcal{J}_1(E, \mu)$ be the ideal of trace class operators $\tilde{K}: L_2(E, \mu) \rightarrow L_2(E, \mu)$ (see volume 1 of [19] for the precise definition); the symbol $\|\tilde{K}\|_{\mathcal{J}_1}$ will stand for the \mathcal{J}_1 -norm of the operator \tilde{K} . Let $\mathcal{J}_2(E, \mu)$ be the ideal of Hilbert-Schmidt operators $\tilde{K}: L_2(E, \mu) \rightarrow L_2(E, \mu)$; the symbol $\|\tilde{K}\|_{\mathcal{J}_2}$ will stand for the \mathcal{J}_2 -norm of the operator \tilde{K} .

Let $\mathcal{J}_{1,\text{loc}}(E, \mu)$ be the space of operators $K: L_2(E, \mu) \rightarrow L_2(E, \mu)$ such that for any bounded Borel subset $B \subset E$ we have

$$\chi_B K \chi_B \in \mathcal{J}_1(E, \mu).$$

Again, we endow the space $\mathcal{J}_{1,\text{loc}}(E, \mu)$ with a countable family of semi-norms

$$(29) \quad \|\chi_B K \chi_B\|_{\mathcal{J}_1}$$

where, as before, B runs through an exhausting family B_n of bounded sets. A locally trace class operator K admits a *kernel*, for which, slightly abusing notation, we use the same symbol K .

2.8. Determinantal Point Processes. A Borel probability measure \mathbb{P} on $\text{Conf}(E)$ is called *determinantal* if there exists an operator $K \in \mathcal{J}_{1,\text{loc}}(E, \mu)$ such that for any bounded measurable function g , for which $g - 1$ is supported in a bounded set B , we have

$$(30) \quad \mathbb{E}_{\mathbb{P}} \Psi_g = \det \left(1 + (g - 1)K\chi_B \right).$$

The Fredholm determinant in (30) is well-defined since $K \in \mathcal{J}_{1,\text{loc}}(E, \mu)$. The equation (30) determines the measure \mathbb{P} uniquely. For any pairwise disjoint bounded Borel sets $B_1, \dots, B_l \subset E$ and any $z_1, \dots, z_l \in \mathbb{C}$ from (30) we have $\mathbb{E}_{\mathbb{P}} z_1^{\#B_1} \dots z_l^{\#B_l} = \det \left(1 + \sum_{j=1}^l (z_j - 1)\chi_{B_j} K\chi_{\cup_i B_i} \right)$.

For further results and background on determinantal point processes, see e.g. [8], [11], [15], [16], [21], [22], [25].

If K belongs to $\mathcal{J}_{1,\text{loc}}(E, \mu)$, then, throughout the paper, we denote the corresponding determinantal measure by \mathbb{P}_K . Note that \mathbb{P}_K is uniquely defined by K , but different operators may yield the same measure. By the Macchi—Soshnikov theorem [17], [25], any Hermitian positive contraction that belongs to the class $\mathcal{J}_{1,\text{loc}}(E, \mu)$ defines a determinantal point process. For the purposes of this paper, we will only be interested in determinantal point processes given by operators of orthogonal projection; there is a standard procedure of doubling the phase space (see e.g. the Appendix in [2]) that reduces the case of contractions to the case of projections.

2.9. The product of a determinantal measure and a multiplicative functional. We start by recalling the results of [6] (see also [5]) showing that the product of a determinantal measure with a multiplicative functional is, after normalization, again a determinantal measure, whose kernel is found explicitly.

Let g be a non-negative measurable function on E . If the operator $1 + (g - 1)K$ is invertible, then we set

$$\mathfrak{B}(g, K) = gK(1 + (g - 1)K)^{-1}, \quad \tilde{\mathfrak{B}}(g, K) = \sqrt{g}K(1 + (g - 1)K)^{-1}\sqrt{g}.$$

By definition, $\mathfrak{B}(g, K), \tilde{\mathfrak{B}}(g, K) \in \mathcal{J}_{1,\text{loc}}(E, \mu)$ since $K \in \mathcal{J}_{1,\text{loc}}(E, \mu)$, and, if K is self-adjoint, then so is $\tilde{\mathfrak{B}}(g, K)$.

We now quote Proposition 2.1 in [5].

Proposition 2.2. *Let $K \in \mathcal{J}_{1,\text{loc}}(E, \mu)$ be a self-adjoint positive contraction, and let \mathbb{P}_K be the corresponding determinantal measure on $\text{Conf}(E)$. Let g be a nonnegative bounded measurable function on E such that*

$$(31) \quad \sqrt{g - 1}K\sqrt{g - 1} \in \mathcal{J}_1(E, \mu),$$

and that the operator $1 + (g - 1)K$ is invertible. Then the operators $\mathfrak{B}(g, K), \tilde{\mathfrak{B}}(g, K)$ induce on $\text{Conf}(E)$ a determinantal measure $\mathbb{P}_{\mathfrak{B}(g, K)} = \mathbb{P}_{\tilde{\mathfrak{B}}(g, K)}$ satisfying

$$(32) \quad \mathbb{P}_{\mathfrak{B}(g, K)} = \frac{\Psi_g \mathbb{P}_K}{\int_{\text{Conf}(E)} \Psi_g d\mathbb{P}_K}.$$

Remark. Of course, from (31) and the invertibility of the operator $1 + (g - 1)K$ we have $\Psi_g \in L_1(\text{Conf}(E), \mathbb{P}_K)$ and

$$\int \Psi_g d\mathbb{P}_K = \det\left(1 + \sqrt{g - 1}K\sqrt{g - 1}\right) > 0,$$

so the right-hand side of (32) is well-defined.

For the reader's convenience, we recall the proof of Proposition 2.2 in the case when the assumption (31) is replaced (cf. [6]) by a simpler assumption

$$(g - 1)K \in \mathcal{J}_1(E, \mu);$$

for the proof in the general case, see [5]. Take a bounded measurable function f on E such that $(f - 1)K \in \mathcal{J}_1(E, \mu)$; for example, one may take f that is different from 1 on a bounded set. We have $(fg - 1)K \in \mathcal{J}_1(E, \mu)$ since $(f - 1)K \in \mathcal{J}_1(E, \mu), (g - 1)K \in \mathcal{J}_1(E, \mu)$. By definition, we have

$$(33) \quad \mathbb{E}_{\mathbb{P}_K} \Psi_f \Psi_g = \det(1 + (fg - 1)K) = \\ = \det(1 + (f - 1)gK(1 + (g - 1)K)^{-1}) \det(1 + (g - 1)K).$$

We rewrite (33) in the form

$$\frac{\mathbb{E}_{\mathbb{P}_K} \Psi_f \Psi_g}{\mathbb{E}_{\mathbb{P}_K} \Psi_g} = \det(1 + (f - 1)\mathfrak{B}(g, K)) = \det(1 + (f - 1)\tilde{\mathfrak{B}}(g, K)).$$

Since a probability measure on the space of configurations is uniquely determined by the values of multiplicative functionals corresponding to all bounded functions f that are different from 1 on a bounded set, formula (33) implies Proposition 2.2.

2.10. Projections and subspaces. Let $L \subset L_2(E, \mu)$ be a closed subspace, let Π be the corresponding projection operator, assumed to be locally of trace class, and let \mathbb{P}_Π the corresponding determinantal measure. Our aim is to determine how the measure \mathbb{P}_Π changes if the subspace L is multiplied by a function.

We start with the following clear

Proposition 2.3. *Let $\alpha(x)$ be a measurable function such that $|\alpha(x)| = 1$ μ -almost surely. Then the operator of orthogonal projection onto the subspace $\alpha(x)L$ induces the same determinantal measure \mathbb{P}_Π .*

Proof. Indeed, if $\Pi(x, y)$ is the kernel of the operator Π , then the kernel of the new operator has the form

$$\frac{\alpha(x)\Pi(x, y)}{\alpha(y)},$$

and such gauge transformations do not change the determinantal measure.

Proposition 2.4. *Let g be a non-negative bounded function on E such that the operator $1 + (g - 1)\Pi$ is invertible. Then the operator*

$$(34) \quad \Pi^g = \sqrt{g}\Pi(1 + (g - 1)\Pi)^{-1}\sqrt{g}$$

is the operator of orthogonal projection onto the closure of the subspace $\sqrt{g}L$.

Proof. First, let $\tilde{\varphi} \in \sqrt{g}L$, that is, $\tilde{\varphi} = \sqrt{g}\varphi$, $\varphi \in L$. Since $\varphi \in L$, we have

$$(1 + (g - 1)\Pi)\varphi = g\varphi,$$

whence

$$(1 + (g - 1)\Pi)^{-1}\sqrt{g}\tilde{\varphi} = \varphi,$$

and finally

$$\Pi^g\tilde{\varphi} = \tilde{\varphi}$$

as desired.

Now take φ to be orthogonal to the subspace $\sqrt{g}L$. Since g is real-valued, we have $\sqrt{g}\varphi \in L^\perp$, whence $(1 + (g - 1)\Pi)^{-1}\sqrt{g}\varphi = \varphi$, whence $\Pi^g\varphi = 0$. The proposition is proved completely.

2.11. Normalized multiplicative functionals. If the multiplicative functional Ψ_g is \mathbb{P}_Π -integrable, then we introduce the *normalized* multiplicative functional $\overline{\Psi}_g$ by the formula

$$(35) \quad \overline{\Psi}_g = \frac{\Psi_g}{\int_{\text{Conf}(E)} \Psi_g d\mathbb{P}_\Pi}.$$

We reformulate Proposition 2.1 in [5] in our new notation (34), (35):

Proposition 2.5. *If g is a bounded Borel function on E such that*

$$\sqrt{g - 1}\Pi\sqrt{g - 1} \in \mathcal{J}_1(E, \mu)$$

and the operator $1 + (g - 1)\Pi$ is invertible, then the normalized multiplicative functional $\overline{\Psi}_g$ is well-defined, the subspace $\sqrt{g}L$ is closed, and we have

$$(36) \quad \overline{\Psi}_g\mathbb{P}_\Pi = \mathbb{P}_{\Pi^g}.$$

Note that closedness of the subspace $\sqrt{g}L$ is immediate from the invertibility of the operator $1 + (g - 1)\Pi$: indeed, the operator $1 + (g - 1)\Pi$ takes the subspace L to the subspace gL , which is consequently closed; since the function g is bounded from above, the subspace $\sqrt{g}L$ is, a fortiori, closed as well.

A key point in the argument of this paper is that the normalized multiplicative functional (35) can be defined, in such a way that the formula (36) still holds, even when the multiplicative functional Ψ_g itself is not defined, see Proposition 4.6 below.

2.12. Palm Measures of Determinantal Point Processes. Palm measures of determinantal point processes admit the following characterization. As above, let $\Pi \in \mathcal{S}_{1,\text{loc}}(E, \mu)$ be the operator of orthogonal projection onto a closed subspace $L \subset L_2(E, \mu)$. For $q \in E$ satisfying $\Pi(q, q) \neq 0$, introduce a kernel Π^q by the formula

$$(37) \quad \Pi^q(x, y) = \Pi(x, y) - \frac{\Pi(x, q)\Pi(q, y)}{\Pi(q, q)}.$$

If $\Pi(q, q) = 0$, then we also have $\Pi(x, q) = \Pi(q, y) = 0$ almost surely with respect to μ , and we set $\Pi^q = \Pi$.

The operator Π^q defines an orthogonal projection onto the subspace

$$L(q) = \{\varphi \in L : \varphi(q) = 0\}$$

of functions in L that assume the value zero at the point q ; the space $L(q)$ is well-defined by Assumption 1; in other words, $L(q)$ is the orthogonal complement of v_q in L . Iterating, let $q_1, \dots, q_l \in E$ be distinct and set

$$L(q_1, \dots, q_l) = \{\varphi \in L : \varphi(q_1) = \dots = \varphi(q_l) = 0\},$$

and let Π^{q_1, \dots, q_l} be the operator of orthogonal projection onto the subspace $L(q_1, \dots, q_l)$. Shirai and Takahashi [21] have proved

Proposition 2.6 (Shirai and Takahashi [21]). *For μ -almost every $q \in E$, the reduced Palm measure $(\mathbb{P}_\Pi)^q$ of the determinantal point process \mathbb{P}_Π at the point q is given by the formula*

$$(38) \quad \mathbb{P}_\Pi^q = \mathbb{P}_{\Pi^q}.$$

Furthermore, for any $l \in \mathbb{N}$ and for ρ_l -almost every l -tuple q_1, \dots, q_l of distinct points in E , the iterated reduced Palm measure $\mathbb{P}_\Pi^{q_1, \dots, q_l}$ is given by the formula

$$(39) \quad \mathbb{P}_\Pi^{q_1, \dots, q_l} = \mathbb{P}_{\Pi^{q_1, \dots, q_l}}.$$

Remark. Shirai and Takahashi [21] have in fact established the formula (38) for arbitrary positive self-adjoint locally trace-class contractions; the

formula (37) for the kernel stays the same. Note that the formula for contractions is a corollary of the formula for projection operators.

2.13. Conditional measures in the discrete case. In this subsection, we consider the discrete case, in which the space E is a countable set endowed with the discrete topology, and the measure μ is the counting measure. In this case, the reduced Palm measure \mathbb{P}^q of a point process \mathbb{P} on $\text{Conf}(E)$ can be described as follows: one takes the conditional measure of \mathbb{P} on the subset of configurations containing a particle at position q , and then one removes the particle at q ; more formally, \mathbb{P}^q is the push-forward of the said conditional measure under the operation that to a configuration X containing the particle at q assigns the configuration $X \setminus \{q\}$.

In the discrete case we also have a dual construction: let $\mathbb{P}^{\check{q}}$ be the conditional measure of \mathbb{P} with respect to the event that there is no particle at position q . More formally, set

$$\text{Conf}(E; E \setminus \{q\}) = \{X \in \text{Conf}(E) : q \notin X\},$$

and write

$$\mathbb{P}^{\check{q}} = \frac{\mathbb{P}|_{\text{Conf}(E; E \setminus \{q\})}}{\mathbb{P}(\text{Conf}(E; E \setminus \{q\}))}$$

be the normalized restriction of \mathbb{P} onto the subset $\text{Conf}(E; E \setminus \{q\})$.

We have a dual to Proposition 2.6.

Proposition 2.7. *Let $q \in E$ be such that $\mu(\{q\}) > 0$. Then the operator of orthogonal projection onto the subspace $\chi_{E \setminus q} L$ has the kernel $\Pi^{\check{q}}$ given by the formula*

$$(40) \quad \Pi^{\check{q}}(x, y) = \Pi(x, y) + \frac{\Pi(x, q)\Pi(q, y)}{1 - \Pi(q, q)}, x \neq q, y \neq q;$$

$$\Pi^{\check{q}}(x, q) = \Pi^{\check{q}}(q, y) = 0, x, y \in E.$$

Proof. This is a particular case of Corollary 6.4 in Lyons [15]; see also Shirai-Takahashi [21], [22].

Given $l \in \mathbb{N}$, $m < l$ and an l -tuple (p_1, \dots, p_l) , of distinct points in E , recall that we have introduced a subspace $L(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$ by the formula

$$(41) \quad L(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l) = \{\chi_{E \setminus \{p_{m+1}, \dots, p_l\}} \varphi : \varphi \in L, \varphi(p_1) = \dots = \varphi(p_l) = 0\}.$$

Let $\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}$ be the operator of orthogonal projection onto the subspace $L(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$. The corresponding determinantal measure $\mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}}$ admits the following characterization. Recall that

$$C(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$$

is the set of configurations on E containing exactly one particle in each of the positions p_1, \dots, p_m and no particles in the positions p_{m+1}, \dots, p_l . There is a natural *erasing bijection* between $\mathbf{C}(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$ and $\mathbf{C}(\check{p}_1, \dots, \check{p}_m, \check{p}_{m+1}, \dots, \check{p}_l)$ obtained by erasing the particles in positions p_1, \dots, p_m .

Proposition 2.8. *Consider the normalized restriction of \mathbb{P}_Π onto the set $\mathbf{C}(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l)$. The push-forward of this normalized restriction onto the set $\mathbf{C}(\check{p}_1, \dots, \check{p}_m, \check{p}_{m+1}, \dots, \check{p}_l)$ under the erasing bijection is the measure $\mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}}$.*

Proof. Again, this is a reformulation of Corollary 6.4 in Lyons [15]; see also Shirai-Takahashi [21], [22].

2.14. Action of Borel automorphisms on point processes. Let $T : E \rightarrow E$ be an invertible measurable map such that for any bounded set $B \subset E$ the set $T(B)$ is also bounded. The map T naturally acts on the space of configurations $\text{Conf}(E)$: namely, given $X \in \text{Conf}(E)$ we set

$$T(X) = \{Tx, x \in X\}$$

Note that, by our assumptions, $T(X)$ is a well-defined configuration on E ; slightly abusing notation, we thus keep the same symbol T for the induced action on the space of configurations.

Let \mathbb{P} be a probability measure on $\text{Conf}(E)$. We assume that \mathbb{P} admits correlation measures of all orders, and, for $l \in \mathbb{N}$, we let ρ_l be the l -th correlation function of the point process \mathbb{P} . The l -th Cartesian power of T naturally acts on the measure ρ_l , and, slightly abusing notation, we denote the resulting measure by $\rho_l \circ T$. The measure $\rho_l \circ T$ is, of course, the l -th correlation measure of the point process $\mathbb{P} \circ T$, the push-forward of the measure \mathbb{P} under the induced action of the automorphism T on the space of configurations.

We now prove a simple general statement: if for a point process \mathbb{P} and all $l \in \mathbb{N}$, the reduced Palm measures corresponding to different l -tuples of points are equivalent, then for any Borel automorphism T acting by the identity beyond a bounded set, the measures \mathbb{P} and $\mathbb{P} \circ T$ are also equivalent, and the Radon-Nikodym derivative is found explicitly in terms of the Radon-Nikodym derivatives of the reduced Palm measures. More precisely, we have the following

Proposition 2.9. *Let $T : E \rightarrow E$ be a Borel automorphism admitting a bounded subset $B \subset E$ such that $T(x) = x$ for all $x \in E \setminus B$. Assume that*

- (1) *for any $l \in \mathbb{N}$, the correlation measures ρ_l and $\rho_l \circ T$ are equivalent;*
- (2) *for any two collections $\{q_1, \dots, q_l\}$ and $\{q'_1, \dots, q'_l\}$ of distinct points of E , the measures $\mathbb{P}^{q_1, \dots, q_l}$ and $\mathbb{P}^{q'_1, \dots, q'_l}$ are equivalent.*

Then the measures \mathbb{P} and $\mathbb{P} \circ T$ on $\text{Conf}(E)$ are equivalent, and for \mathbb{P} -almost every configuration $X \in \text{Conf}(E)$ such that $X \cap B = \{q_1, \dots, q_l\}$ we have

$$\frac{d\mathbb{P} \circ T}{d\mathbb{P}}(X) = \frac{d\mathbb{P}^{Tq_1, \dots, Tq_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(X \setminus \{q_1, \dots, q_l\}) \times \frac{d\rho_l \circ T}{d\rho_l}(q_1, \dots, q_l).$$

Proposition 2.9 is a particular case of the following general proposition on absolute continuity of point processes. Let $l \in \mathbb{N}$, let $\mathbb{P}, \tilde{\mathbb{P}}$ be probability measures on $\text{Conf}(E)$ admitting correlation functions of order l . Let $\rho_l, \tilde{\rho}_l$ be the corresponding correlation measures, $\mathbb{P}^{q_1, \dots, q_l}, \tilde{\mathbb{P}}^{q_1, \dots, q_l}$ are corresponding reduced Palm measures. The symbol \ll denotes absolute continuity of measures.

Proposition 2.10. *If $\rho_l \ll \tilde{\rho}_l$ and $\tilde{\mathbb{P}}^{q_1, \dots, q_l} \ll \mathbb{P}^{q_1, \dots, q_l}$ for ρ_l -almost any distinct $q_1, \dots, q_l \in E$, then also $\tilde{\mathbb{P}} \ll \mathbb{P}$ and for \mathbb{P} -almost any $X \in \text{Conf}(E)$ and any l particles $q_1, \dots, q_l \in X$ we have*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{d\tilde{\mathbb{P}}^{q_1, \dots, q_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(X \setminus \{q_1, \dots, q_l\}) \times \frac{d\tilde{\rho}_l}{d\rho_l}(q_1, \dots, q_l).$$

Proposition 2.10, in turn, is immediate from Proposition 2.1: indeed, take $X_0 \in \text{Conf}(E)$ and $q_1^{(0)}, \dots, q_l^{(0)} \in X_0$; take disjoint bounded open sets $B^{(1)}, \dots, B^{(l)} \subset E$, set $B = \cup B^{(i)}$ and take an open set $U \subset E$ disjoint from all $B^{(i)}$ in such a way that $q_i(0) \in B^{(i)}$ for all $i = 1, \dots, l$ and $X_0 \setminus \{q_1^{(0)}, \dots, q_l^{(0)}\} \subset U$. Let \mathcal{W} be a neighbourhood of $X_0 \setminus \{q_1^{(0)}, \dots, q_l^{(0)}\}$ in $\text{Conf}(E)$ satisfying $\mathcal{W} \subset \{X \in \text{Conf}(E) : X \subset U\}$. Introduce a neighbourhood \mathcal{Z} of X_0 by setting

$$(42) \quad \mathcal{Z} = \{X \in \text{Conf}(E) : \#_{B^{(1)}}(X) = \dots = \#_{B^{(l)}}(X) = 1, X|_{E \setminus B} \subset \mathcal{W}\}.$$

Sets given by (42) form a basis of neighbourhoods of X_0 .

By definition of Palm measures and Proposition 2.1, we have

$$\mathbb{P}(\mathcal{Z}) = \int_{B^{(1)} \times \dots \times B^{(l)}} \mathbb{P}^{q_1, \dots, q_l}(\mathcal{Z}) d\rho_l(q_1, \dots, q_l).$$

A similar formula holds for $\tilde{\mathbb{P}}$, and Proposition 2.10 is proved.

We now derive Proposition 2.9 from Proposition 2.10. As before, let $\text{Conf}(E; E \setminus B)$ be the subset of those configurations on E all whose particles lie in $E \setminus B$. since the automorphism T acts by the identity on $E \setminus B$, all configurations in the set $\text{Conf}(E; E \setminus B)$ are fixed by T , and we have

$$\mathbb{P}^{q_1, \dots, q_l}|_{\text{Conf}(E; E \setminus B)} \circ T = \mathbb{P}^{Tq_1, \dots, Tq_l}|_{\text{Conf}(E; E \setminus B)}.$$

Since, in the notation of Proposition 2.9, we have

$$X \setminus \{q_1, \dots, q_l\} \in \text{Conf}(E; E \setminus B),$$

we also have

$$\frac{d\mathbb{P}^{q_1, \dots, q_l} \circ T}{d\mathbb{P}^{q_1, \dots, q_l}}(X \setminus \{q_1, \dots, q_l\}) = \frac{d\mathbb{P}^{Tq_1, \dots, Tq_l}}{d\mathbb{P}^{q_1, \dots, q_l}}(X \setminus \{q_1, \dots, q_l\}),$$

and Proposition 2.9 follows now from Proposition 2.10. Proposition 2.9 is proved completely.

3. INTEGRABILITY AND CONDITIONING

3.1. Integrability of the Palm kernel.

Lemma 3.1. *Let $q \in U$ be such that $\Pi(q, q) \neq 0$. Then the kernel of the operator Π^q has the integrable form*

$$(43) \quad \Pi^q(x, y) = \frac{A^q(x)B^q(y) - A^q(y)B^q(x)}{x - y}$$

where

$$(44) \quad A^q(x) = \frac{A(x)B(q) - A(q)B(x)}{\sqrt{(A(q))^2 + (B(q))^2}},$$

$$B^q(x) = \frac{A(x)A(q) + B(x)B(q)}{\sqrt{(A(q))^2 + (B(q))^2}} - \frac{\sqrt{(A(q))^2 + (B(q))^2}(A(x)B(q) - A(q)B(x))}{\Pi(q, q)(x - q)}.$$

Proof. We first consider the case

$$A(q) = 0, B(q) \neq 0.$$

Then

$$\Pi(x, q) = \frac{A(x)B(q)}{x - q}$$

and

$$\Pi^q(x, y) = \Pi(x, y) - \frac{B(q)^2 A(x)A(y)}{\Pi(q, q)(x - q)(y - q)} = \frac{A^q(x)B^q(y) - A^q(y)B^q(x)}{x - y}$$

with

$$A^q(x) = A(x), B^q(x) = B(x) - \frac{B(q)^2 A(x)}{\Pi(q, q)(x - q)},$$

as desired. The general case is reduced to this particular one by a rotation

$$A(x) \rightarrow \frac{A(x)B(q) - A(q)B(x)}{\sqrt{(A(q))^2 + (B(q))^2}};$$

$$B(x) \rightarrow \frac{A(x)A(q) + B(x)B(q)}{\sqrt{(A(q))^2 + (B(q))^2}}.$$

The proposition is proved.

3.2. The relation between Palm subspaces.

3.2.1. The subspace L' .

Proposition 3.2. *If $\varphi \in L_2(\mathbb{R}, \mu)$ is such that $x\varphi \in L_2(\mathbb{R}, \mu)$, then the integrals*

$$\int_{\mathbb{R}} \varphi(x)A(x)d\mu(x), \int_{\mathbb{R}} \varphi(x)B(x)d\mu(x)$$

are well-defined.

The proof is immediate from the clear relation

$$\langle v_p(x), (x-p)\varphi(x) \rangle = A(p) \int_{\mathbb{R}} \varphi(x)B(x)d\mu(x) - B(p) \int_{\mathbb{R}} \varphi(x)A(x)d\mu(x).$$

Let

(45)

$$L' = \{\psi \in L : x\psi \in L_2(\mathbb{R}, \mu), \int_{\mathbb{R}} \psi(x)A(x)d\mu(x) = \int_{\mathbb{R}} \psi(x)B(x)d\mu(x) = 0\}.$$

Proposition 3.3. *Let $p \in U$ and $\varphi \in L$ satisfy $\varphi(p) = 0$. Then there exists $\psi \in L'$ such that*

$$(46) \quad \varphi(x) = (x - p)\psi(x).$$

Proof. It suffices to consider the case $p = 0$, $A(0) = 0$, $B(0) \neq 0$: the general case is reduced to this one by a translation of \mathbb{R} and a linear unimodular change of variable (5).

Let ψ' be such that (46) holds (in the continuous case, such a function ψ' is unique; in the discrete case, however, there are many such functions, differing by their value at $p = 0$). Applying the commutator $x\Pi - \Pi x$ to the function ψ' , we obtain

$$(47) \quad x\Pi\psi'(x) - \varphi(x) = \\ = A(x) \int_{\mathbb{R}} B(y)\psi'(y)d\mu(y) - B(x) \int_{\mathbb{R}} A(y)\psi'(y)d\mu(y).$$

Since $\varphi \in L$, $\varphi(0) = 0$, $A(0) = 0$, $B(0) \neq 0$, substituting $x = 0$ into (47) we obtain

$$(48) \quad \int_{\mathbb{R}} A(y)\psi'(y)d\mu(y) = 0.$$

Recall that by definition we have

$$v_0(x) = \frac{A(x)}{x} \in L.$$

Dividing (47) by x and keeping (48) in mind, we obtain that there exists $\alpha \in \mathbb{C}$ such that

$$(49) \quad \Pi\psi'(x) - \psi(x) - \alpha\delta_0 = v_0(x) \int_{\mathbb{R}} B(y)\psi'(y)d\mu(y).$$

The extra term $\alpha\delta_0$ is only necessary in the case when $\mu(0) > 0$. It follows that we have $\psi = \psi' + \alpha\delta_0 \in L$ and, consequently, applying the commutator $x\Pi - \Pi x$ to the function ψ , that we also have

$$\int_{\mathbb{R}} A(y)\psi(y)d\mu(y) = \int_{\mathbb{R}} B(y)\psi(y)d\mu(y) = 0.$$

The proposition is proved completely.

We proceed with the proofs of Theorems 1.5, 1.7. We must now separately consider the case of continuous and the case of purely atomic measures μ .

3.2.2. The case of continuous measures. Assume that the measure μ satisfies $\mu(\{p\}) = 0$ for any $p \in E$ and, as before, let Π be a locally trace class operator with an integrable kernel defined on an open subset $U \subset \mathbb{R}$ whose complement has measure 0.

3.2.3. The relation between Palm subspaces.

Proposition 3.4. *For any distinct points $p_1, \dots, p_l, q_1, \dots, q_l \in U$ we have*

$$(50) \quad L(p_1, \dots, p_l) = \frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)} L(q_1, \dots, q_l).$$

Remark. The coincidence of subspaces is understood as coincidence of subspaces in L_2 ; the functions from the right-hand side subspace are of course not defined at the points q_1, \dots, q_l ; they are nonetheless well-defined as elements of L_2 since the measure μ is continuous. For discrete measures the fomulation will be modified.

Proof. The proof proceeds by induction on l . We start with $l = 1$. By Proposition 3.3, we have

$$\frac{x - p_1}{x - q_1} L(q_1) \subset L(q_1) + L' \subset L.$$

By definition, now, any function belonging to the subspace $\frac{x - p_1}{x - q_1} L(q_1)$ assumes value 0 at the point p_1 , whence the inclusion

$$\frac{x - p_1}{x - q_1} L(q_1) \subset L(p_1).$$

Interchanging the points p_1 and q_1 , we obtain the converse inclusion (using again continuity of the measure μ), and the proposition is proved for $l = 1$.

By Lemma 3.1, Palm measures of determinantal point processes given by projection operators with integrable kernels are themselves given by projection operators with integrable kernels. Applying, step by step, Proposition 3.4 for $l = 1$, we obtain

$$\begin{aligned}
 (51) \quad L(p_1, \dots, p_l) &= \frac{(x - p_1)}{(x - q_1)} L(q_1, p_2, \dots, p_l) = \\
 &= \frac{(x - p_1)(x - p_2)}{(x - q_1)(x - q_2)} L(q_1, q_2, \dots, p_l) = \dots \\
 &= \frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)} L(q_1, \dots, q_l),
 \end{aligned}$$

and Proposition 3.4 is proved completely.

3.2.4. The case of discrete measures. We now proceed to the case of atomic measures; without losing generality, we assume that the set E is countable and the measure μ is the counting measure. Since we are only interested in determinantal measures, we need only consider configurations without multiple points: in other words, the space $\text{Conf}(E)$ can be identified with the space of infinite binary sequences.

In the discrete case, we have a dual to Proposition 3.1.

Proposition 3.5. *Let Π be a projection operator with an integrable kernel. Let $q \in U$ be such that $\Pi(q, q) \neq 1$. Then the kernel of the operator $\Pi^{\check{q}}$ has the integrable form*

$$(52) \quad \Pi^{\check{q}}(x, y) = \frac{A^{\check{q}}(x)B^{\check{q}}(y) - A^{\check{q}}(y)B^{\check{q}}(x)}{x - y}$$

where $A^{\check{q}}(q) = B^{\check{q}}(q) = 0$ and for $x \neq q, y \neq q$ we have

$$\begin{aligned}
 (53) \quad A^{\check{q}}(x) &= \frac{A(x)B(q) - A(q)B(x)}{\sqrt{(A(q))^2 + (B(q))^2}}, \\
 B^{\check{q}}(x) &= \frac{A(x)A(q) + B(x)B(q)}{\sqrt{(A(q))^2 + (B(q))^2}} + \frac{\sqrt{(A(q))^2 + (B(q))^2}(A(x)B(q) - A(q)B(x))}{(1 - \Pi(q, q))(x - q)}.
 \end{aligned}$$

Proof. This is a straightforward verification using Proposition 2.7 and proceeding in the same way as the proof of Proposition 3.1. As in Proposition 3.1, the integrable representation for the kernel is, of course, far from unique.

Proposition 3.6. *Let the kernel Π be integrable. Let $p_1, \dots, p_l \in E$ be distinct, and let π be a permutation of $\{1, \dots, l\}$. Then we have*

$$(54) \quad L(p_{\pi(1)}, \dots, p_{\pi(m)}, \check{p}_{\pi(m+1)}, \dots, \check{p}_{\pi(l)}) = \\ = \chi_{E \setminus \{p_1, \dots, p_l\}}(x) \frac{(x - p_{\pi(1)}) \dots (x - p_{\pi(m)})}{(x - p_1) \dots (x - p_m)} L(p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l).$$

Proof. As in the continuous case, we proceed by induction and start with the case $l = 2, m = 1$: we need to show, for any distinct $p, q \in E$, the equality

$$(55) \quad L(p, \check{q}) = \chi_{E \setminus \{p, q\}} \frac{x - p}{x - q} L(q, \check{p}).$$

Now, by Proposition 3.3, we have

$$\chi_{E \setminus q} \frac{L(q)}{x - q} \subset \chi_{E \setminus q} L.$$

Since

$$\frac{x - p}{x - q} = 1 + \frac{q - p}{x - q},$$

we also have

$$\frac{x - p}{x - q} \chi_{E \setminus q} L(q) \subset \chi_{E \setminus q} L = L(\check{q})$$

Now, multiplying any function by $\chi_{E \setminus q} \frac{x - p}{x - q}$ yields a function that assumes value 0 at the point p ; we thus conclude

$$(56) \quad \chi_{E \setminus \{p, q\}} \frac{x - p}{x - q} L(\check{p}, q) \subset L(p, \check{q}).$$

Interchanging the variables p, q , we obtain the inverse inclusion, and (55) is proved.

We proceed with the induction argument. Our kernel Π is integrable. We know from Proposition 3.1 that for any $q \in E$ the kernel Π^q is also integrable, while Proposition 3.5 implies that the kernel $\Pi^{\check{q}}$ is integrable as well. The conclusion of Proposition 3.6 in the particular case $l = 2, m = 1$ is thus valid for any kernel of the form $\Pi^{q_1, \dots, q_r, \check{q}_{r+1}, \dots, \check{q}_s}$ with $r \leq s$ and q_1, \dots, q_s arbitrary distinct points in E . Representing the permutation π as a product of transpositions and applying (56) repeatedly, we conclude the proof of Proposition 3.6.

4. MULTIPLICATIVE FUNCTIONALS AND REGULARIZATION

The main result of this Section is Proposition 4.6 below; before formulating it, we make some preliminary remarks.

4.1. An estimate of diagonal values of the kernel Π^g . As the following proposition shows, diagonal values of the kernel of Π^g can be estimated from above by the diagonal values of the kernel Π .

Proposition 4.1. *Let the kernel Π satisfy Assumption 1, and let g be a non-negative bounded function on E such that the operator $1 + (g - 1)\Pi$ is invertible. Then for any $q \in U$ we have*

$$\Pi^g(q, q) \leq g(q) \|(1 + (g - 1)\Pi)^{-1}\| \Pi(q, q).$$

Proof. As before, we let $\langle \cdot, \cdot \rangle$ be the standard inner product in $L_2(E, \mu)$ and we write $v_q(x) = \Pi(x, q)$ so that $\Pi(q, q) = \langle v_q, v_q \rangle$. By definition then

$$(57) \quad \begin{aligned} \Pi^g(q, q) &= g(q) \langle \Pi(1 + (g - 1)\Pi)^{-1} v_q, v_q \rangle \leq \\ &\leq g(q) \|(1 + (g - 1)\Pi)^{-1}\| \langle v_q, v_q \rangle, \end{aligned}$$

and the proposition is proved.

4.2. Multiplicative functionals corresponding to a function satisfying $g \leq 1$. Let g be a nonnegative function satisfying $g \leq 1$. If

$$(58) \quad \|\chi_{\{x \in E: g(x) < 1\}} \Pi\| < 1,$$

then the space $\sqrt{g}L$ is closed and the operator $1 + (g - 1)\Pi$ is invertible. Indeed, the assumption (58) together with the inequality $0 \leq g \leq 1$ immediately implies the inequality $\|(g - 1)\Pi\| < 1$. Invertibility of the operator $1 + (g - 1)\Pi$ is established, and since this operator takes the subspace L to the subspace gL , it follows that the subspace gL is closed, whence, a fortiori, since $0 \leq g \leq 1$, the subspace $\sqrt{g}L$ is also closed. Summing up, we obtain

Proposition 4.2. *Let g be a bounded measurable function on E satisfying $g \leq 1$ and (58). If*

$$(59) \quad \text{tr} \left(\chi_{\{x \in E: g(x) < 1\}} \Pi \chi_{\{x \in E: g(x) < 1\}} \right) < +\infty,$$

then all the conclusions of Proposition 2.5 hold for the function g .

Remark. The condition (58) can be equivalently reformulated as follows. Let $E_1 = \{x \in E : g(x) = 1\}$, $E_2 = \{x \in E : g(x) < 1\}$. The requirement (58) is equivalent to the existence of a positive constant C such that for any $\varphi \in L$ we have

$$\|\chi_{E_2} \varphi\| \leq C \|\chi_{E_1} \varphi\|.$$

Consequently, if the condition (58) holds for the subspace L , then it also holds for any subspace of the form hL , where h is a positive function bounded above and bounded away from 0.

4.3. Multiplicative functionals corresponding to a function g satisfying $g \geq 1$.

4.3.1. *The case when the function g is bounded.* Proposition 2.5 takes a simpler form when our bounded function g satisfies $g \geq 1$. First, in this case the subspace $\sqrt{g}L$ is automatically closed. Second, if $\sqrt{g-1}\Pi\sqrt{g-1}$ belong to the trace class, then the operator $1 + (g-1)\Pi$ is automatically invertible. To verify this, observe first that in this case the operator $\sqrt{g-1}\Pi$ is Hilbert-Schmidt, consequently, so is $(g-1)\Pi$. To check the invertibility of the operator $1 + (g-1)\Pi$, it thus suffices to prove that a function φ satisfying

$$(60) \quad \varphi + (g-1)\Pi\varphi = 0$$

must be the zero function. Set $\psi = -\sqrt{g-1}\Pi\varphi$ so that $\varphi = \sqrt{g-1}\psi$. Note that both φ and ψ are by definition zero on the set $\{x \in E : g(x) = 1\}$. From (60) we now have

$$\psi + \sqrt{g-1}\Pi\sqrt{g-1}\psi = 0,$$

whence

$$\langle \psi, \psi \rangle + \langle \Pi\varphi, \varphi \rangle = 0,$$

whence finally $\varphi = \psi = 0$.

We can now reformulate Proposition 2.5 in the following simpler form

Proposition 4.3. *Let g be a bounded measurable function on E satisfying $g \geq 1$ and such that the operator $\sqrt{g-1}\Pi\sqrt{g-1}$ belongs to the trace class. Then all the conclusions of Proposition 2.5 hold for the function g .*

4.3.2. *The case of unbounded g .* The function $(x-p)/(x-q)$ is unbounded on \mathbb{R} , and we prepare, for future use, a proposition on multiplicative functionals corresponding to unbounded functions.

Proposition 4.4. *Let g be a measurable function on E satisfying $g \geq 1$. Assume that*

- (1) *the space $\sqrt{g}L$ is a closed subspace of $L_2(E, \mu)$;*
- (2) *the operator $\sqrt{g-1}\Pi$ is Hilbert-Schmidt;*
- (3) *the operator Π^g of orthogonal projection onto the space $\sqrt{g}L$ is locally of trace class, and there exists $R > 0$ such that*

$$(61) \quad \text{tr} \left(\chi_{\{x \in E : g(x) > R\}} \Pi^g \chi_{\{x \in E : g(x) > R\}} \right) < +\infty.$$

Then $\Psi_g \in L_1(\text{Conf}(E), \mathbb{P}_\Pi)$, and the corresponding normalized multiplicative functional $\overline{\Psi}_g$ satisfies

$$(62) \quad \overline{\Psi}_g \mathbb{P}_\Pi = \mathbb{P}_{\Pi^g}.$$

Remark. Of course, the subspace $\sqrt{g}L$, provided it lies in L_2 , is automatically closed.

Proof. The Hilbert-Schmidt norm of the operator $\sqrt{g-1}\Pi$ is given by the expression

$$\int_E (g(x) - 1) \Pi(x, x) d\mu(x).$$

Convergence of this integral implies, in particular, for any $R > 1$, the estimate

$$\text{tr} \left(\chi_{\{x \in E: g(x) > R\}} \Pi \chi_{\{x \in E: g(x) > R\}} \right) < +\infty,$$

whence it follows that the function g is bounded on \mathbb{P}_Π -almost every configuration.

For $R > 0$ set $g^R(x) = g(x)$ if $g(x) < R$ and $g^R(x) = 1$ otherwise. Since the operator $\sqrt{g-1}\Pi$ is Hilbert-Schmidt, the operator $\sqrt{g^R-1}\Pi$ is a fortiori Hilbert-Schmidt, and we have

$$\int_{\text{Conf}(E)} \Psi_{g^R} d\mathbb{P}_\Pi = \det(1 + \sqrt{g^R-1}\Pi\sqrt{g^R-1}).$$

By definition and since the function g is bounded on \mathbb{P}_Π -almost every configuration, we have the \mathbb{P}_Π -almost sure convergence

$$\Psi_g = \lim_{R \rightarrow \infty} \Psi_{g^R}.$$

The determinant in the right-hand side is bounded above by a constant depending only on the Hilbert-Schmidt norm of $\sqrt{g-1}\Pi$, and integrability of Ψ_g is thus established.

It remains to check the equality (62). Let R be big enough in such a way that

$$(63) \quad \text{tr} \left(\chi_{\{x \in E: g(x) > R\}} \Pi^g \chi_{\{x \in E: g(x) > R\}} \right) < 1.$$

Set $\tilde{g} = g^R/g$. We clearly have $\Psi_{\tilde{g}} \in L_1(\text{Conf}(E), \mathbb{P}_{\Pi^g})$, and since $\sqrt{\tilde{g}}(\sqrt{g}L) = \sqrt{g^R}L$, Proposition 4.2 implies the relation

$$\overline{\Psi}_{\tilde{g}} \mathbb{P}_{\Pi^g} = \mathbb{P}_{\Pi^g R}.$$

By definition, $\Psi_{\tilde{g}} \Psi_g = \Psi_{g^R}$. Since we already know that

$$\overline{\Psi}_{g^R} \mathbb{P}_\Pi = \mathbb{P}_{\Pi^g R},$$

the equality (62) is proved, and Proposition 4.4 is proved completely.

4.4. Regularization of additive functionals. Let $f : E \rightarrow \mathbb{C}$ be a Borel function. We set S_f to be the corresponding additive functional, and, if $S_f \in L_1(\text{Conf}(E), \mathbb{P}_\Pi)$, then we set

$$(64) \quad \overline{S}_f = S_f - \mathbb{E}S_f.$$

The random variable \overline{S}_f will be called the *normalized* additive functional corresponding to f . We shall now see that the normalized additive functional can be defined even when the additive functional itself is not well-defined. Set

$$\text{Var}(\Pi, f) = \frac{1}{2} \int_E \int_E |f(x) - f(y)|^2 |\Pi(x, y)|^2 d\mu(x) d\mu(y).$$

Note that the value $\text{Var}(\Pi, f)$ does not change if the function f is changed by an additive constant. If $S_f \in L_2(\text{Conf}(E), \mathbb{P}_\Pi)$, then $\text{Var}(\Pi, f) < +\infty$ and

$$(65) \quad \text{Var}(S_f) = \mathbb{E}|\overline{S}_f|^2 = \text{Var}(\Pi, f).$$

Note also the clear inequality

$$(66) \quad \text{Var}(\Pi, f) \leq 2 \int_E |f(x)|^2 \Pi(x, x) d\mu(x)$$

which is obtained by summing the inequality $|f(x) - f(y)|^2 \leq 2(|f(x)|^2 + |f(y)|^2)$ over all x, y and using the Pythagoras theorem

$$\Pi(x, x) = \int_E |\Pi(x, y)|^2 d\mu(y).$$

These formulae show that the integral defining the variance of an additive functional may converge even when the integral defining its expectation does not. The normalized additive functional can thus by continuity be defined in L_2 even when the additive functional itself diverges almost surely.

We therefore introduce the Hilbert space $\mathcal{V}(\Pi)$ in the following way: the elements of $\mathcal{V}(\Pi)$ are functions f on E satisfying $\text{Var}(\Pi, f) < +\infty$; functions that differ by a constant are identified, but, slightly abusing terminology we still refer to elements of $\mathcal{V}(\Pi)$ as functions. The square of the norm of an element $f \in \mathcal{V}(\Pi)$ is precisely $\text{Var}(\Pi, f)$. By definition, bounded functions that are identically zero in the complement of a bounded set form a dense subset of $\mathcal{V}(\Pi)$. The correspondence $f \rightarrow \overline{S}_f$ is thus an isometric embedding of a dense subset of $\mathcal{V}(\Pi)$ into $L_2(\text{Conf}(E), \mathbb{P}_\Pi)$; it therefore admits a unique isometric extension onto the whole space $\mathcal{V}(\Pi)$, and we obtain the following

Proposition 4.5. *There exists a unique linear isometric embedding*

$$\overline{S} : \mathcal{V}(\Pi) \rightarrow L_2(\text{Conf}(E), \mathbb{P}_\Pi), \quad \overline{S} : f \rightarrow \overline{S}_f$$

such that

- (1) $\mathbb{E}\overline{S}_f = 0$ for all $f \in \mathcal{V}(\Pi)$;
- (2) if $S_f \in L_1(\text{Conf}(E), \mathbb{P}_\Pi)$, then \overline{S}_f is given by (64).

4.5. Regularization of multiplicative functionals. Given a function g such that $\text{Var}(\Pi, \log g) < +\infty$, set

$$\tilde{\Psi}_g = \exp(\overline{S}_{\log g}).$$

By definition, we have

$$\overline{\Psi}_{g_1 g_2} = \overline{\Psi}_{g_1} \overline{\Psi}_{g_2}.$$

Since $\mathbb{E}\overline{S}_{\log g} = 0$, by Jensen's inequality, for any positive function g we have

$$\mathbb{E}\tilde{\Psi}_g \geq 1.$$

The expectation $\mathbb{E}\tilde{\Psi}_g$ may however be infinite, and our next aim is to give conditions for its finiteness.

It will be convenient for us to allow zero values for the function g : let therefore g be nonnegative, set $E_0 = \{x \in E : g(x) = 0\}$, assume that the subset $\text{Conf}(E; E \setminus E_0)$ of those configurations all whose particles lie in $E \setminus E_0$ has positive probability, consider the restriction of our measure \mathbb{P} onto the subspace $\text{Conf}(E; E \setminus E_0)$, introduce the corresponding functional $\tilde{\Psi}_g$ and extend it to the whole of E by setting $\tilde{\Psi}_g(X) = 0$ for all configurations containing a particle at E_0 . Assume that $\text{tr}\chi_{E_0}\Pi\chi_{E_0} < +\infty$. Then we have $\mathbb{P}_\Pi(\text{Conf}(E; E \setminus E_0)) = \det(1 - \chi_{E_0}\Pi\chi_{E_0})$. In particular, $\mathbb{P}_\Pi(\text{Conf}(E; E \setminus E_0)) > 0$ provided that the following holds: if $\varphi \in L$ satisfies $\varphi(x) = 0$ for all $x \in E \setminus E_0$, then $\varphi = 0$ identically.

If $\tilde{\Psi}_g \in L_1(\text{Conf}(E), \mathbb{P}_\Pi)$, then, as before, we write

$$\overline{\Psi}_g = \frac{\tilde{\Psi}_g}{\mathbb{E}\tilde{\Psi}_g}.$$

The main result of this section is

Proposition 4.6. *Let $E_0 \subset E$ be a Borel subset satisfying $\text{tr}\chi_{E_0}\Pi\chi_{E_0} < +\infty$ and such that if $\varphi \in L$ satisfies $\varphi(x) = 0$ for all $x \in E \setminus E_0$, then $\varphi = 0$ identically.*

Let g be a nonnegative function, positive on $E \setminus E_0$ and admitting $\varepsilon > 0$ such that the set $E_\varepsilon = \{x \in E : |g(x) - 1| > \varepsilon\}$ is bounded and

$$(67) \quad \int_{E_\varepsilon} |g(x) - 1| \Pi(x, x) d\mu(x) + \int_{\mathbb{R} \setminus E_\varepsilon} |g(x) - 1|^2 \Pi(x, x) d\mu(x) < +\infty.$$

Then $\tilde{\Psi}_g \in L_1(\text{Conf}(E), \mathbb{P}_\Pi)$. If the subspace $\sqrt{g}L$ is closed and the corresponding operator of orthogonal projection Π^g satisfies, for a sufficiently large R , the condition

$$(68) \quad \text{tr} \left(\chi_{\{x \in E: g(x) > R\}} \Pi^g \chi_{\{x \in E: g(x) > R\}} \right) < +\infty,$$

then we also have

$$(69) \quad \mathbb{P}_{\Pi^g} = \overline{\Psi}_g \mathbb{P}_\Pi.$$

As we shall see below, Propositions 1.4, 1.6 are immediate Corollaries of Proposition 4.6.

Proof. First, our assumptions imply $\mathbb{P}_\Pi(\text{Conf}(E; E \setminus E_0)) > 0$ and so, without losing generality, restricting ourselves, if necessary, to the subset $\text{Conf}(E; E \setminus E_0)$, we can assume that the function g is positive (observe here that, by Proposition 4.1, applied to the function $\chi_{E \setminus E_0}$, the condition (67) continues to hold for the restricted kernel).

We will need a simple auxiliary

Lemma 4.7. *If f is a bounded function on E , then the Hilbert-Schmidt norm $\|f\Pi f\|_2$ of the operator $f\Pi f$ satisfies*

$$(70) \quad \|f\Pi f\|_2 \leq \int_E (f(x))^2 \Pi(x, x) d\mu(x).$$

Proof. Indeed, the Cauchy-Bunyakovsky-Schwarz inequality implies

$$(\Pi(x, y))^2 \leq \Pi(x, x) \Pi(y, y),$$

whence

$$\|f\Pi f\|_2^2 = \int_E \int_E f(x)^2 f(y)^2 \Pi(x, y)^2 d\mu(x) d\mu(y) \leq \left(\int_E f(x)^2 \Pi(x, x) d\mu(x) \right)^2.$$

The proof is complete. We proceed to the proof of Proposition 4.6 and first consider the case of bounded functions. Let $\mathcal{A}_2(\Pi)$ be the set of positive Borel functions g on E satisfying

$$(1) \quad \infty > \sup_E g \geq \inf_E g > 0;$$

$$(2)$$

$$\int_E |g(x) - 1|^2 \Pi(x, x) d\mu(x) < +\infty.$$

By definition, the set $\mathcal{A}_2(\Pi)$ is a semigroup under multiplication.

Endow the set $\mathcal{A}_2(\Pi)$ with a metric by setting the distance between two functions g_1 and g_2 to be

$$\sqrt{\int_E |g_1(x) - g_2(x)|^2 \Pi(x, x) d\mu(x)}.$$

Using the second condition in the definition of $\mathcal{A}_2(\Pi)$ and the estimate (66), for any $g \in \mathcal{A}_2(\Pi)$ we have

$$\text{Var}(\Pi, g - 1) < +\infty.$$

Since on any interval of the positive half-line, bounded away from zero and infinity, the quantity $|\log t - t + 1|/t^2$ is bounded both above and below, for any function $g \in \mathcal{A}_2(\Pi)$, we also have

$$\text{Var}(\Pi, \log g) < +\infty.$$

In particular, for any function $g \in \mathcal{A}_2(\Pi)$ the functional $\tilde{\Psi}_g$ is well-defined. Our next aim is to establish its integrability.

Proposition 4.8. *For any function $g \in \mathcal{A}_2(\Pi)$ we have $\tilde{\Psi}_g \in L_1(\text{Conf}(E), \mathbb{P}_\Pi)$. The correspondences*

$$g \rightarrow \tilde{\Psi}_g, g \rightarrow \overline{\Psi}_g$$

are continuous mappings from $\mathcal{A}_2(\Pi)$ to $L_1(\text{Conf}(E), \mathbb{P}_\Pi)$.

First we give an upper bound for the L_2 -norm of $\tilde{\Psi}_g$.

Proposition 4.9. *For any $\varepsilon > 0$, $M > 0$ there exists a constant $C_{\varepsilon, M} > 0$ such that if $g \in \mathcal{A}_2(\Pi)$ satisfies*

$$(71) \quad M \geq \sup_E g \geq \inf_E g \geq \varepsilon$$

then

$$(72) \quad \log \mathbb{E} |\tilde{\Psi}_g|^2 \leq C_{\varepsilon, M} \int_E |g(x) - 1|^2 \Pi(x, x) d\mu(x);$$

Proof. It suffices to prove the estimate

$$(73) \quad \log \mathbb{E} \tilde{\Psi}_g \leq C_{\varepsilon, M} \int_E |g(x) - 1|^2 \Pi(x, x) d\mu(x),$$

and (72) follows by multiplicativity (perhaps with a different constant). It suffices to establish (73) in the case when the set $\{x \in E : g(x) \neq 1\}$ is

bounded, as the general case follows by Fatou's lemma. Now there exists a constant $C_2 > 0$ such that

$$(74) \quad \begin{aligned} \log \mathbb{E}\Psi_g &\leq \text{tr}(\sqrt{g-1}\Pi\sqrt{g-1}) + C_2 \|\sqrt{g-1}\Pi\sqrt{g-1}\|_2^2 \leq \\ &\leq \int_E (g(x) - 1)\Pi(x, x)d\mu(x) + C_2 \int_E |g(x) - 1|^2 \Pi(x, x)d\mu(x). \end{aligned}$$

Note that we have assumed boundedness of the set $\{x \in E : g(x) \neq 1\}$ in order that the integral

$$\int_E (g(x) - 1)\Pi(x, x)d\mu(x)$$

be well-defined; this integral will, however, disappear from the final result. Indeed, from (71), again using the fact that the quantity $|\log t - t + 1|/t^2$ is bounded both above and below by constants only depending on ε and M , we obtain

$$(75) \quad \begin{aligned} \left| \int_E (g(x) - 1)\Pi(x, x)d\mu(x) - \int_E \log g(x)\Pi(x, x)d\mu(x) \right| &\leq \\ &\leq C_{M,\varepsilon} \int_E |g(x) - 1|^2 \Pi(x, x)d\mu(x), \end{aligned}$$

whence

$$\log \mathbb{E}\tilde{\Psi}_g = \log \mathbb{E}\Psi_g - \mathbb{E}S_{\log g} \leq C'_{M,\varepsilon} \int_E |g(x) - 1|^2 \Pi(x, x)d\mu(x),$$

and the proposition is proved.

Proposition 4.10. *For any $\varepsilon > 0$, $M > 0$ there exists a constant $C_{\varepsilon,M} > 0$ such that if $g_1, g_2 \in \mathcal{A}_2(\Pi)$ satisfy*

$$M \geq \sup_E g_1 \geq \inf_E g_1 \geq \varepsilon, M \geq \sup_E g_2 \geq \inf_E g_2 \geq \varepsilon,$$

then

$$\mathbb{E}|\tilde{\Psi}_{g_1} - \tilde{\Psi}_{g_2}| \leq \mathbb{E}|\tilde{\Psi}_{g_1}|^2 \left(\exp \left(C_{\varepsilon,M} \int_E |g_1(x) - g_2(x)|^2 \Pi(x, x)d\mu(x) \right) - 1 \right).$$

Proof. Since $\mathbb{E}\tilde{\Psi}_g \geq 1$, we have

$$\mathbb{E}|\tilde{\Psi}_g - 1|^2 \leq \mathbb{E}\tilde{\Psi}_{g^2} - 1.$$

From the estimate (72) we have

$$(76) \quad \mathbb{E}|\tilde{\Psi}_g - 1|^2 \leq \exp \left(C \int_E |g(x) - 1|^2 \Pi(x, x) d\mu(x) \right) - 1.$$

Applying (76) to $g = g_1/g_2$, recalling the boundedness of both g_1 and g_2 and using multiplicativity, we obtain the proposition.

Proposition 4.10 implies Proposition 4.8.

We next check that the regularized multiplicative functional corresponding to a function g is indeed the Radon-Nikodym derivative of the measure \mathbb{P}_{Π^g} with respect to the measure \mathbb{P}_Π .

First note that if $g \in \mathcal{A}_2(\Pi)$, then the subspace $\sqrt{g}L$ is automatically closed; as always, we set Π^g to be the corresponding operator of orthogonal projection.

Corollary 4.11. *Let $g \in \mathcal{A}_2(\Pi)$ satisfy*

$$\sup_{x \in E} |g(x) - 1| < 1.$$

Then the operator Π_g is locally of trace class, and we have

$$(77) \quad \mathbb{P}_{\Pi^g} = \overline{\Psi}_g \mathbb{P}_\Pi.$$

Proof. By our assumptions, we have $\|(g - 1)\Pi\| < 1$. The operator

$$\Pi^g = \sqrt{g}\Pi(1 + (g - 1)\Pi)^{-1}\sqrt{g}$$

is locally of trace class since so is Π . Let $E^{(n)}$ be a sequence of bounded sets exhausting E , and set $g_n = 1 + (g - 1)\chi_{E^{(n)}}$. We have

$$\sup_n \|(g_n - 1)\Pi\| < 1.$$

The operators

$$\Pi^{g_n} = \sqrt{g_n}\Pi(1 + (g_n - 1)\Pi)^{-1}\sqrt{g_n}$$

are by definition locally of trace class since so is Π . As $n \rightarrow \infty$, we have the strong operator convergence

$$(1 + (g_n - 1)\Pi)^{-1} \rightarrow (1 + (g - 1)\Pi)^{-1}$$

and, consequently, also the convergence

$$\Pi^{g_n} \rightarrow \Pi^g$$

in the space of locally trace class operators. It follows that, as $n \rightarrow \infty$, the sequence of measures $\mathbb{P}_{\Pi^{g_n}}$ weakly converges to \mathbb{P}_{Π^g} in the space of probability measures on $\text{Conf}(E)$.

Furthermore, $\overline{\Psi}_{g_n} \rightarrow \overline{\Psi}_g$ in $L_1(\text{Conf}(E), \mathbb{P}_\Pi)$, whence also

$$\overline{\Psi}_{g_n} \mathbb{P}_\Pi \rightarrow \overline{\Psi}_g \mathbb{P}_\Pi$$

weakly in the space of probability measures on $\text{Conf}(E)$. We finally obtain the desired equality (77).

4.6. Conclusion of the proof of Proposition 4.6. Choose $\varepsilon \in (0, 1)$ in such a way that we have

$$(78) \quad \|\chi_{\{x \in E: g(x) < 1-\varepsilon\}} \Pi\| < 1.$$

Set

$$(79) \quad g_1 = (g - 1)\chi_{\{x \in E: |g(x)-1| \leq \varepsilon\}} + 1.$$

$$(80) \quad g_2 = (g - 1)\chi_{\{x \in E: g(x) < 1-\varepsilon\}} + 1.$$

$$(81) \quad g_3 = (g - 1)\chi_{\{x \in E: g(x) > 1+\varepsilon\}} + 1.$$

By definition, $g = g_1 g_2 g_3$.

By definition, the subspace $\sqrt{g_1}L$ is closed, and, by Corollary 4.11, we have

$$\mathbb{P}_{\Pi^{g_1}} = \overline{\Psi}_{g_1} \mathbb{P}_{\Pi}.$$

Proposition 4.1 implies the existence of a positive constant C such that

$$\Pi^{g_1}(x, x) \leq C \Pi(x, x)$$

for μ -almost all $x \in E$.

Next, the inequality (78) implies that all the assumptions of Proposition 4.2 are verified for the function g_2 and the operator Π^{g_1} (cf. Remark following the proof of Proposition 4.2); applying Proposition 4.2 to the function g_2 and the operator Π^{g_1} , we arrive at the formula

$$\mathbb{P}_{\Pi^{g_1 g_2}} = \overline{\Psi}_{g_2} \mathbb{P}_{\Pi^{g_1}} = \overline{\Psi}_{g_1 g_2} \mathbb{P}_{\Pi}.$$

Again, Proposition 4.1 implies the existence of a positive constant C such that

$$(82) \quad \Pi^{g_1 g_2}(x, x) \leq C \Pi(x, x)$$

for μ -almost all $x \in E$.

In the third step, we apply Proposition 4.4 to the function g_3 and the operator $\Pi^{g_1 g_2}$. In order to be able to do so, we first verify, one by one, the assumptions of Proposition 4.4 for the function g_3 and the operator $\Pi^{g_1 g_2}$. First, the subspace $\sqrt{g_3}(\sqrt{g_1 g_2})L = \sqrt{g}L$ is closed. Second, the assumption (67) of Proposition 4.6, together with the estimate (82), implies the estimate

$$\int_E |g_3(x) - 1| \Pi^{g_1 g_2}(x, x) d\mu(x) < +\infty,$$

and, consequently, that the operator $\sqrt{g_3 - 1}\Pi^{g_1g_2}$ is Hilbert-Schmidt. Finally, from (68), keeping in mind that $(\Pi^{g_1g_2})^{g_3} = \Pi^g$, we immediately have, for sufficiently large R , the desired estimate

$$\mathrm{tr} \left(\chi_{\{x \in E: g_3(x) > R\}} (\Pi^{g_1g_2})^{g_3} \chi_{\{x \in E: g_3(x) > R\}} \right) < +\infty.$$

Applying Proposition 4.4 to the function g_3 and the operator $\Pi^{g_1g_2}$, we have

$$\mathbb{P}_{\Pi^{g_1g_2g_3}} = \overline{\Psi}_{g_3} \mathbb{P}_{\Pi^{g_1g_2}}.$$

Observe that we only used regularized multiplicative functionals at the very first step of our argument. In other words, there exist constants C_1, C_2, C_3 such that we have

$$\begin{aligned} \mathbb{P}_{\Pi^{g_1}} &= C_1 \tilde{\Psi}_{g_1} \mathbb{P}_{\Pi}, \\ \mathbb{P}_{\Pi^{g_1g_2}} &= C_2 \Psi_{g_2} \mathbb{P}_{\Pi^{g_1}}, \\ \mathbb{P}_{\Pi^g} &= \mathbb{P}_{\Pi^{g_1g_2g_3}} = C_3 \Psi_{g_3} \mathbb{P}_{\Pi^{g_1g_2}}. \end{aligned}$$

By definition, we have

$$\tilde{\Psi}_g = \tilde{\Psi}_{g_1} \tilde{\Psi}_{g_2} \tilde{\Psi}_{g_3}$$

and, consequently, for a suitable positive constant C_4 , also

$$\tilde{\Psi}_g = C_4 \tilde{\Psi}_{g_1} \Psi_{g_2} \Psi_{g_3}.$$

Summing up, we finally obtain

$$\mathbb{P}_{\Pi^g} = \overline{\Psi}_g \mathbb{P}_{\Pi}.$$

Proposition 4.6 is proved completely.

4.7. Proof of Proposition 1.4. We check that for $l \in \mathbb{N}$ and any distinct points $p_1, \dots, p_l, q_1, \dots, q_l \in U$, the function

$$g(x) = \left(\frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)} \right)^2$$

satisfies

- (1) $\mathrm{Var}(\log g, \Pi^{q_1, \dots, q_l}) < +\infty$;
- (2) For any bounded interval $I \subset \mathbb{R}$ we have

$$\int_I g(x) d\Pi^{q_1, \dots, q_l}(x, x) d\mu(x) < +\infty;$$

- (3) For any $\varepsilon > 0$, we have

$$(83) \quad \int_{\{x \in \mathbb{R}: \min_{i=1, \dots, l} |x - q_i| > \varepsilon\}} |g(x) - 1|^2 \Pi^{q_1, \dots, q_l}(x, x) d\mu(x) < +\infty.$$

Remark. Of course, convergence (83) for some $\varepsilon > 0$ is equivalent to convergence (83) for any $\varepsilon > 0$.

Indeed, since the kernel Π is smooth, for any $i = 1, \dots, l$, as x ranges in a sufficiently small neighbourhood of point q_i , we have

$$|\Pi^{q_1, \dots, q_l}(x, x)| < C|x - q_i|^2.$$

Consequently, the integrand $g(x)\Pi^{q_1, \dots, q_l}(x, x)$ is bounded on I , and the second condition follows. The third condition is an immediate corollary of (6): indeed, on the set $\{x \in \mathbb{R} : \min_{i=1, \dots, l} |x - q_i| > \varepsilon\}$ we have

$$|g(x) - 1|^2 \leq \text{const} \times \frac{1}{1 + x^2},$$

with the constant depending only on q_1, \dots, q_l and ε . It follows that

$$\int_{\{x \in \mathbb{R} : \min_{i=1, \dots, l} |x - q_i| > \varepsilon\}} |g(x) - 1|^2 \Pi(x, x) d\mu(x) < +\infty,$$

and, since the operator Π^{q_1, \dots, q_l} is a finite-rank perturbation of the operator Π , the desired inequality follows as well.

The second and the third conclusion together imply

$$(84) \quad \int_{\mathbb{R}} |\log g(x)|^2 d\Pi^{q_1, \dots, q_l}(x, x) d\mu(x) < +\infty$$

(indeed, for small values of g integrability in (84) follows from the second condition, for large values of g from the third.) The first conclusion follows from (84). In particular, all assumptions of Proposition 4.6 are satisfied for the function g and the kernel Π^{q_1, \dots, q_l} and the normalized multiplicative functional $\bar{\Psi}_g$ is well-defined with respect to the measure $\mathbb{P}_{\Pi^{q_1, \dots, q_l}}$. The proposition is proved completely.

Proposition 4.6 together with Proposition 3.4 imply the following immediate

Corollary 4.12. *Under the assumptions of Theorem 1.5, for any distinct points $p_1, \dots, p_l, q_1, \dots, q_l \in U$, for the corresponding reduced Palm measures are equivalent, and we have*

$$\frac{d\mathbb{P}_{\Pi^{p_1, \dots, p_l}}}{d\mathbb{P}_{\Pi^{q_1, \dots, q_l}}} = \bar{\Psi} \left| \frac{(x - p_1) \dots (x - p_l)}{(x - q_1) \dots (x - q_l)} \right|^2.$$

Together with Proposition 2.9, Corollary 4.12 implies Theorem 1.5. Theorem 1.5 is proved completely.

4.8. Proof of Proposition 1.6. Denote $q_i = \sigma(p_i)$, $i = 1, \dots, l$; of course, we have $\{p_1, \dots, p_l\} = \{q_1, \dots, q_l\}$. Apply Proposition 4.6 to the function

$$(85) \quad g(x) = \prod_{i=1}^m \left(\frac{x - q_i}{x - p_i} \right)^2 \chi_{E \setminus \{p_1, \dots, p_l\}}(x).$$

The function g is bounded, and the condition (9) implies

$$\sum_E |g(x) - 1|^2 < +\infty,$$

which, in turn, immediately implies (67). Since the subspace L does not contain functions supported on finite sets, all other assumptions of Proposition 4.6 are verified for the kernel $\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}$ and the function g given by (85). Proposition 1.6 is proved completely.

In a similar way to the continuous case, Proposition 4.6 and Proposition 3.6 together imply that, under the assumptions of Proposition 1.6, we have

$$(86) \quad \mathbb{P}_{\Pi^{q_1, \dots, q_m, \check{q}_{m+1}, \dots, \check{q}_l}} = \overline{\Psi}(p_1, \dots, p_l, m, \sigma) \mathbb{P}_{\Pi^{p_1, \dots, p_m, \check{p}_{m+1}, \dots, \check{p}_l}}.$$

The relation (86) together with Proposition 2.9 implies Theorem 1.7.

Theorem 1.7 is proved completely.

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